Lawvere's Law as an Elimination Principle for Identity Types

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February 26, 2021
CMU HoTT Student Workshop

Path Induction *vs.* Lawvere's Law

Awodey (2017)

"The elimination rule [of the identity type] is a form of what may be called 'Lawvere's Law'"

$$\frac{z : A \vdash R(z, z, \mathsf{refl}_z)}{x, y : A, p : x =_A y \vdash R(x, y, p)} \text{ Path Induction}$$

$$\frac{z : A \vdash R(z, z)}{x, y : A, p : x =_A y \vdash R(x, y)} \text{ Lawvere's Law}$$

Lawvere's Law as Path Recursion

$$\frac{z : A \vdash R(z, z, \mathsf{refl}_z)}{x, y : A, p : x =_A y \vdash R(x, y, p)} \text{PI}$$

$$\frac{z : A \vdash R(z, z)}{x, y : A, p : x =_A y \vdash R(x, y)} \text{LL}$$

Formally, Lawvere's Law is the corresponding recursion principle to path induction, and we will also use the term "**path recursion**".

Motivation: Lawvere's Law in Logic

In logic, Lawvere's Law characterizes the equality predicate by **adjointness**, as the "least reflexive relation".

$$\frac{\top_A \vdash R(z,z)}{x =_A y \vdash R(x,y)} \, \mathsf{LL}$$

$$\frac{\top_A \vdash R(x,y)[z/x, z/y]}{x =_A y \vdash R(x,y)} \, \mathsf{LL}$$

$$\frac{\top_A \vdash R(x,y)[\Delta]}{\exists_\Delta(\top_A) \vdash R(x,y)} \, \mathsf{LL}$$

,

where $\Delta : A \to A \times A$ is the diagonal.

Result

The purpose of the talk is to prove the following:

Theorem

Lawvere's Law (path recursion with propositional η) is equivalent to path induction.

So, the identity type can be given by its adjointness property.

The proof applies methods known from Escardó (2015) and Hofmann (1990s).

Outline

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Conventions

We will work with the following setting and conventions:

- dependent type theory with Π- and Σ-types;
- rules are stated with unaffected contexts, substitution bookkeeping, *etc.*, implicit;
- the usual formation and introduction rules for the $=$-type, *i.e.*

$$\frac{}{x, y : A \vdash x =_A y \ \mathrm{Type}} =\text{-F} \qquad \frac{}{z : A \vdash \mathrm{refl}_z : z =_A z} =\text{-I} \quad .$$

Path Induction

We recall the usual rules for path induction (PI).

$$\frac{z : A \vdash d(z) : R(z, z, \mathsf{refl}_z)}{x, y : A, p : x =_A y \vdash j(z.d, x, y, p) : R(x, y, p)}\ \text{PI-E}$$

$$\frac{z : A \vdash d(z) : R(z, z, \mathsf{refl}_z)}{z : A \vdash j(z.d, z, z, \mathsf{refl}_z) \equiv d(z) : R(z, z, \mathsf{refl}_z)}\ \text{PI-}\beta$$

Path Recursion (Lawvere's Law)

For us, path recursion (PR) will include the usual rules for path recursion, together with rules for propositional η-conversion. We start with the usual rules for path recursion.

$$\frac{z : A \vdash d(z) : R(z,z)}{x, y : A, p : x =_A y \vdash j(z.d, x, y, p) : R(x,y)}\ \text{PR-E}$$

$$\frac{z : A \vdash d(z) : R(z,z)}{z : A \vdash j(z.d, z, z, \mathrm{refl}_z) \equiv d(z) : R(z,z)}\ \text{PR-}\beta$$

η-Conversion

We further assume two rules controlling η-conversion as part of PR.

$$\frac{x, y : A, p : x =_A y \vdash t(x,y,p) : R(x,y)}{'' \vdash \rho(x.y.p.t, x, y, p) : t(x,y,p) = j(z.t(z,z,\mathsf{refl}_z), x, y, p)}\ \text{PR-}\eta$$

$$\frac{x, y : A, p : x =_A y \vdash t(x,y,p) : R(x,y)}{z : A \vdash \rho(x.y.p.t, z, z, \mathsf{refl}_z) \equiv \mathsf{refl}_{t(z,z,\mathsf{refl}_z)}}\ \text{PR-}\eta\text{-Coh.}$$

The rule PR-η-Coh. is, to some degree, optional, but we will use it to derive the definitional PI-β.

Outline

PI iff PR

We are now in a position to restate the theorem precisely.

Theorem

The rule sets PI and PR are interderivable.

The forward direction is trivial. The rest of the current section shows the backward direction.

Usual Proofs from PR

In many cases, the usual proofs using PI (*i.e.* PI-E and PI-β) do not use power in excess of PR-E and PR-β. We thus obtain:

Lemma

PR implies

- *symmetry and transitivity of propositional equality;*
- *transport.*

Consider, *e.g.*, the case of symmetry. The term

$$x, y : A, p : x =_A y \vdash p^{-1} : y =_A x$$

is usually constructed by path induction, but the type $y =_A x$ has no path-dependence.

Contractibility at a Point

In order to derive PI-β, we will use a strong form of contractibility.

Definition

We will say that a type A is contractible at $a : A$ *when*

- *a is an ordinary center of contraction for A (explicitly, we have $\rho(x) : a =_A x$, for all $x : A$);*
- *the contraction of a to itself is* refl_a *(explicitly, $\rho(a) \equiv \mathsf{refl}_a$).*

Contractibility of Based Path Space (Statement)

Definition (BPSC)

When, for any type A and $x : A$, $\sum_{\alpha:A} x =_A \alpha$ is contractible at (x, refl_x), we say that based path spaces are contractible at the base point. *We denote this property by* BPSC*.*

Lemma

PR implies BPSC.

This is our key lemma, and the method is due to Escardó (2015).

Contractibility of Based Path Space (Proof)

Proof.

For contractibility, it suffices to show that $(x, \mathsf{refl}_x) = (y, p)$, for all $x, y : A, p : x =_A y$.
By PR-η, we have

$$(y, p) = \ell(z.(z, \mathsf{refl}_z), x, y, p)$$

and

$$(x, \mathsf{refl}_x) = \ell(z.(z, \mathsf{refl}_z), x, y, p) \quad .$$

By symmetry and transitivity, we then have $(x, \mathsf{refl}_x) = (y, p)$, as desired.
By PR-η-Coh., this construction exhibits $\sum_{\alpha:A} x =_A \alpha$ as contractible at (x, refl_x). □

Contractibility of Based Path Space (Remark)

Remark

The type $\sum_{\alpha:A} x =_A \alpha$ did not depend on $y : A$, so the applications of PR-η were somewhat degenerate.

Deriving PI

The following lemma is well-known:

Lemma (Hofmann)

Transport and BPSC together imply PI.

Sketch.

It suffices to derive based path induction. When

$$x, y : A, p : x =_A y \vdash R(x, y) \qquad \text{and} \qquad x : A \vdash d(x) : R(x, x),$$

we derive

$$x : A, \gamma : \textstyle\sum_{y:A} x =_A y \vdash R(x, p_1(\gamma)) \quad \text{and} \quad x : A \vdash d(x) : R(x, p_1(x, \mathrm{refl}_x)).$$

We have a contraction $(x, \mathrm{refl}_x) = (y, p)$, yielding the transport

$$\mathrm{trans}(x, y, p, d(x)) : R(x, p_1(y, p)) \equiv R(x, y) \qquad .$$

$\square$

Reprise of Theorem

We thus conclude the proof of the main result:

Theorem

The rule sets PI and PR are interderivable.

Outline

Summary

	Symmetric	Asymmetric
Path-Dependency	Path Induction	Based Path Induction
No Path-Dependency	Path Recursion (LL) (Diagonal Yoneda)	Based Path Recursion ('Based LL') (Rijke's Yoneda)

Figure: Summary of Equivalent Elimination Principles for Identity Types

Leibniz's Law

Definition (Leibniz's Law)

Leibniz's Law *will denote the principle*

$$\prod_{A:U} \prod_{x,y:A} x =_A y \simeq \prod_{P:A \to U} P(x) \simeq P(y)$$

What is the relation to PI? (Ladyman and Presnell 2017)

Thanks for your attention!

References

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