A Yoneda Lemma for synthetic fibered ∞-categories

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Outline

Overview

- **Goal:** A Yoneda Lemma for presheaves *of ∞-categories*, in type theory
- What is an ∞-category? “Some kind of object where directed arrows can be composed weakly.”
- **Setting:** ∞-categories in HoTT? Work in Riehl–Shulman's simplicial HoTT so ∞-categories become definable internally (basic objects are *simplicial* types; also suggested by Joyal)
- **Result:** Define fibered ∞-categories and prove fibered Yoneda Lemma à la Riehl–Verity, Street, Riehl–Shulman (discrete case).
- **Interpretation:** Yoneda Lemma as *directed* arrow induction principle.

Outline

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman I

Simplicial type theory, introduced by Riehl–Shulman [RS17], as an extension of HoTT (cf. cubical type theory):

1. **n-dimensional directed cubes (*cube layer*):** Lawvere theory generated by cube $\mathbb{2}$:

$$\mathbf{1}, \mathbb{2}, \mathbb{2} \times \mathbb{2}, \ldots, \mathbb{2}^n, \ldots$$

2. **Subpolytopes of n-cubes (*tope layer*):** Coherent (without $\exists$) intuitionistic theory of formulas in cube contexts with strict equality judgments $\equiv$, and inequality $\leq$ on $\mathbb{2}$
3. **Import into ordinary HoTT (*shape layer*):** Shape = Cube together with a tope. Types can depend on shapes.

$$\frac{I \,\mathrm{cube} \qquad t : I \vdash \varphi \,\mathrm{tope}}{\{t : I \mid \varphi\} \,\mathrm{shape}}$$

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman II

Some important shapes:

Δ^1

$$0 \longrightarrow 1$$

Δ^2

$$\begin{tikzcd}
\langle 0,1 \rangle & \langle 1,1 \rangle \\
\langle 0,0 \rangle \arrow[ur] \arrow[r] & \langle 1,0 \rangle \arrow[u] \arrow[ul, phantom, "="]
\end{tikzcd}$$

$\partial\Delta^2$

$$\begin{tikzcd}
\langle 0,1 \rangle & \langle 1,1 \rangle \\
\langle 0,0 \rangle \arrow[ur] \arrow[r] & \langle 1,0 \rangle \arrow[u]
\end{tikzcd}$$

Λ^2_1

$$\begin{tikzcd}
\langle 0,1 \rangle & \langle 1,1 \rangle \\
\langle 0,0 \rangle \arrow[r] & \langle 1,0 \rangle \arrow[u]
\end{tikzcd}$$

$$\Delta^1 :\equiv \{t : \mathbb{2} \mid \top\}, \quad \Delta^2 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \leq t\},$$
$$\partial\Delta^2 :\equiv \{t : \mathbb{2}^2 \mid t \equiv s \vee s \equiv 0 \vee t \equiv 1\}, \quad \Lambda^2_1 :\equiv \{\langle t, s\rangle : \mathbb{2}^2 \mid (s \equiv 0) \vee (t \equiv 1)\}$$

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman III

As a main feature, STT provides *extension types* (after Lumsdaine–Shulman), i.e. for shape inclusions $\Phi \rightarrowtail \Psi$, families $P : \Psi \to \mathcal{U}$, and partial sections $\sigma : \prod_{t:\Phi} P(t)$ there exists the type of sections

$$\left\langle \prod_{t:\Psi} P(t) \middle|^{\Phi}_{\sigma} \right\rangle \triangleq \left\{ \begin{tikzcd} \Phi \arrow[r, "\sigma"] \arrow[d, tail] & P \\ \Psi \arrow[ur, dashed, "\bar{\sigma}"'] & \end{tikzcd} \right\}$$

judgmentally extending a. If $\tau : \langle \prod_{t:\Psi} P(t) |^{\Phi}_{\sigma} \rangle$, then $\tau|_{\Phi} \equiv \sigma$.
An axiom is added to ensure the extension types are homotopically well-behaved.
Furthermore, we can coerce shapes to be fibrant, and we can prove that *weak/homotopical extensions* can always be strictified, *i.e.* we have an equivalence

$$\left\langle \prod_{(x:I|\psi)} A(x) \middle|^{\varphi}_{a} \right\rangle \simeq \sum_{f:\prod_{(x:I|\psi)} A(x)} \prod_{(x:I|\varphi)} (a\,x = f\,x).$$

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman IV

Definition (Hom types, [RS17])

Let B be a type. Fix terms $a, b : B$. The type of *arrows in* B *from* a *to* b is the extension type

$$\hom_B(a,b) :\equiv (a \to b) :\equiv \left\langle \Delta^1 \to B \middle|^{\partial\Delta^1}_{[a,b]} \right\rangle .$$

Definition (Dependent hom types, [RS17])

Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \hom_B(a,b)$ in B and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *arrows in* P *over* u *from* d *to* e is the extension type

$$\mathrm{dhom}_{P,u}(d,e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \middle|^{\partial\Delta^1}_{[d,e]} \right\rangle .$$

Similarly for 2-simplices or other shapes.

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman V

Definition (Synthetic ∞-categories, [RS17])

- *Synthetic ∞-precategory aka Segal type:* type A such that $(\Delta^2 \to A) \xrightarrow{\simeq} (\Lambda_1^2 \to A)$ (Joyal).
- *Synthetic ∞-category aka Rezk type:* type A such that
$$\mathrm{idtoiso}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y).$$
- *Synthetic ∞-groupoid aka discrete type:* type A such that
$$\mathrm{idtoarr}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \hom_A(x,y).$$

Foundations: Synthetic $(\infty, 1)$-categories à la Riehl–Shulman VI

Segal types are synthetic ∞-*precategories*, *i.e.* types with weak composition of morphisms:

$$\mathrm{isSegal}(B) \simeq \prod_{\kappa:\Lambda_1^2 \to B} \mathrm{isContr}\left(\left\langle \Delta^2 \to B \Big|_{\kappa}^{\Lambda_1^2} \right\rangle \right)$$

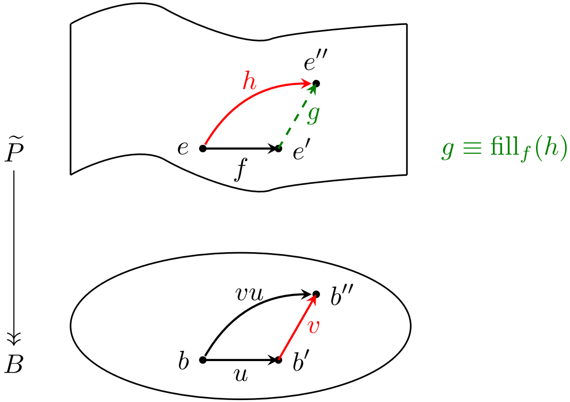
$$\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\parallel"] & \\ a \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & {} & c \end{tikzcd}$$

In particular, they do have categorical structure (associative composition of morphisms, identites, and the corresponding laws).
Maps between Segal types are automatically functors (cf. [RS17]).

Outline

Cocartesian arrows I

An arrow $f : e \to_u e'$ over $u : b \to b'$ is *cocartesian* if it satisfies the following universal property:



Cocartesian arrows II

Fix a map between Rezk types.

Proposition

1. *Being a cocartesian arrow is a proposition.*
2. *Cocartesian lifts are unique up to homotopy.*

Proposition

Let $f : e \to e'$ and $g : e' \to e''$ be arrows in E. Then the following statements hold:

1. *If f, g are cocartesian then so is gf (closedness under composition).*
2. *If f and gf are cocartesian then so is g (right cancelation).*
3. *Cocartesian lifts of identity morphisms are identity morphisms.*

Cocartesian families

Definition (Cocartesian family)

Let B be a Rezk type and $P : B \to \mathcal{U}$ be a family such that $\widetilde{P}$ is a Rezk type. Then P is a *cocartesian family* if:

$$\mathrm{hasCocartLifts}\, P :\equiv \prod_{b,b':B} \prod_{u:b\to b'} \prod_{e:P\,b} \sum_{e':P\,b'} \sum_{f:e\to_u e'} \mathrm{isCocartArr}_P\, f$$

$$\begin{tikzcd} e \arrow[r, dashed, "P_*(u,e)"] & u_* e \\ b \arrow[r, "u"'] & b' \end{tikzcd}$$

Notation:

$$\longrightarrow\!\!\!\!\!\!\!\!\!\!+\!\!\!\!\!\!+\!\!\!\!\!\!+\!\!\!\!\longrightarrow$$

Closure properties

Cocartesian families (resp., fibrations) are closed under: composition, products, reindexing/pullback, ...

$$
\begin{tikzcd}
F \arrow[d, two heads, "\xi"] \arrow[dd, two heads, bend right, "\pi\circ\xi"'] \\
E \arrow[d, two heads, "\pi"] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
F \times E \arrow[d, two heads, "\xi\times\pi"] \\
A \times B
\end{tikzcd}
\qquad
\begin{tikzcd}
j^*E \simeq \sum_{c:C} P(jc) \arrow[r] \arrow[d, two heads, "j^*\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \simeq \sum_{b:B} P(b) \arrow[d, two heads, "\pi"] \\
C \arrow[r, "j"'] & B
\end{tikzcd}
$$

Functoriality of cocartesian families

Cocartesian families are *functorial* w.r.t. arrows in the base:

$$a \xrightarrow{u} b \qquad \rightsquigarrow \qquad Pa \xrightarrow{u_*} Pb$$

with $u_*(d) :\equiv \partial_1 P_*(u, d)$. The induced functors are natural in the sense that

$$(\mathrm{id}_a)_* = \mathrm{id}_{Pa}, \quad (vu)_* = v_* u_*$$

for $v : \hom_B(b, c)$.

$$\begin{tikzcd}
\widetilde{P} \arrow[d, two heads] & P\,a \arrow[r, bend left, "u_*"] \arrow[rr, bend left=40, "(vu)_* = v_* u_*"] & P\,b \arrow[r, bend left, "v_*"] & P\,c \\
B & a \arrow[r, "u"] & b \arrow[r, "v"] & c
\end{tikzcd}$$

The codomain family I

Example: Codomain family

$$\begin{tikzcd} B^{\Delta^1} \arrow[d, two heads, "\partial_1"'] & a \arrow[d, "v"'] \\ & b \\ B & b \end{tikzcd} \quad \rightsquigarrow \quad \begin{array}{c} B \xrightarrow{\mathrm{arrto}_B} \mathcal{U} \\ \mathrm{arrto}_B(b) :\equiv \sum_{a:B} \hom_B(a,b) \end{array}$$

The codomain family II

Example: Codomain family

$$\begin{tikzcd}
B^{\Delta^1} \arrow[ddd, two heads, "\partial_1"'] & a \arrow[r, equal] \arrow[dd, "v"'] & a \arrow[dd, "uv"] \\
 & & \\
 & b \arrow[r, "u"'] & b' \\
B & b \arrow[r, "u"'] & b'
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{array}{c}
B \xrightarrow{\mathrm{arrto}_B} \mathcal{U} \\[2em]
\mathrm{arrto}_B(b) :\equiv \sum_{a:B} \hom_B(a, b)
\end{array}$$

Outline

The classical Yoneda Lemma

The Classical **Yoneda Lemma**[1] is a fundamental principle in category theory: Let $\mathbb{C}$ be a small category. Then, given a functor $F : \mathbb{C} \to \mathrm{Set}$, for any $I : \mathbb{C}$, the following induced map is a (natural) bijection:

$$\mathrm{Nat}(\mathbf{Y}I, F) \xrightarrow{\simeq} F(I), \quad \varphi \mapsto \varphi_I(\mathrm{id}_I)$$

Here, $\mathbf{Y}I$ denotes the *Yoneda Functor*

$$\begin{aligned} \mathbf{Y}I &: \mathbb{C} \to \mathrm{Set}, \\ \mathbf{Y}I(X) &:= \hom_{\mathbb{C}}(I, X), \\ \mathbf{Y}I(f : X \to X') &:= (f \circ - : \hom_{\mathbb{C}}(I, X) \to \hom_{\mathbb{C}'}(I, X')). \end{aligned}$$

[1] covariant version

The classical Yoneda Lemma, fibrationally I

The copresheaves have associated projections:

$$\begin{tikzcd} \widetilde{F} \simeq \sum_{X:\mathbb{C}} F(X) \arrow[d, "\pi_F"'] \\ \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} I \downarrow \mathbb{C} \simeq \sum_{X:\mathbb{C}} \hom_{\mathbb{C}}(I, X) \arrow[d, "\partial_1"'] \\ \mathbb{C} \end{tikzcd}$$

$$F : \mathbb{C} \to \mathrm{Set} \qquad \mathbf{Y}I : \mathbb{C} \to \mathrm{Set}$$

The evaluation map becomes a fiberwise map:

$$\begin{tikzcd} I \downarrow \mathbb{C} \arrow[rr, "\widetilde{\varphi}"] \arrow[dr, "\partial_1"'] & & \widetilde{F} \arrow[dl, "\pi_F"] \\ & \mathbb{C} & \end{tikzcd} \qquad \Big(\prod_{X:\mathbb{C}} \hom_{\mathbb{C}}(I, X) \to F(X)\Big) \simeq \prod_{u:I\downarrow\mathbb{C}} F(\partial_1 u)$$

The classical Yoneda Lemma, fibrationally II

Then the Yoneda Lemma reads:

$$\prod_{I:\mathbb{C}} \mathrm{isEquiv}\Big(\prod_{u:I\downarrow\mathbb{C}} F(\partial_1 u) \xrightarrow{\lambda\sigma.\sigma(\mathrm{id}_I)} F(I)\Big)$$

In the context of synthetic ∞-categories this has been proven by Riehl–Shulman for the discrete case, *i.e.* functorial families of ∞-groupoids. What about functorial families of ∞-categories, *i.e.* cocartesian families? The desired statement is:

Theorem (Yoneda Lemma for cocartesian families, [RV21], Thm. 5.7.3)

Let $P : B \to \mathcal{U}$ be a cocartesian family over a Rezk type B. Then:

$$\prod_{b:B} \mathrm{isEquiv}\Big(\prod_{u:b\downarrow B}^{\mathrm{cocart}} P(\partial_1 u) \xrightarrow{\mathrm{ev}_{\mathrm{id}_b}} P(b)\Big)$$

The classical Yoneda Lemma, fibrationally III

It is necessary to restrict to *cocartesian sections*. A section $\sigma : \prod_{b:B} Pb$ is cocartesian if for any arrow $u : b \to b'$ in B the induced arrow $\sigma u : \sigma b \to_u \sigma b'$ is cocartesian.

Proof of the Yoneda Lemma I

Definition (Initial element)

Let B be a type. A term $b : B$ is *initial* if

$$\prod_{a:B} \mathrm{isContr}(\hom_B(b,a)).$$

For $x : B$, we denote the homotopically uniquely determined map by $\emptyset_x : b \to x$.

Step 1: Define a map in the converse direction:

$$\begin{tikzcd} \prod_B P \arrow[rr, bend right, "\mathrm{ev}_b"'] & & P\,b \arrow[ll, bend right, "\mathbf{y}"'] \end{tikzcd} \qquad \mathbf{y}d :\equiv \lambda x.(\emptyset_x)_*(d)$$

Proof of the Yoneda Lemma II

Geometrically the map $\mathbf{y}$ acts as follows:

$$\begin{tikzcd} d \arrow[rr, "P_*(\emptyset_x, d)", "\shortmid\!\shortmid\!\shortmid"'] & & \mathbf{y}d(x) \\ b \arrow[rr, dashed, "\emptyset_x"] & & x \end{tikzcd}$$

Step 2:

Proposition (cf. [RV21], Ch. 5)

The map $\mathbf{y} : P\, b \to \prod_B P$ *is valued in cocartesian sections, i.e. :*

$$\prod_{u:B} \mathrm{isCocartArr}_P((\mathbf{y}d)u)$$

Proof of the Yoneda Lemma III

Proof.

For $d : P\, b$, consider the constant map $\mathrm{cst}(d) :\equiv \lambda x.d : B \to E$. Define a natural transformation of functors $B \to E$:

$$\mathrm{cst}(d) \xRightarrow{\chi} \mathbf{y}d \qquad\qquad d \xrightarrow[\text{+\!\!\!+}]{\chi_x :\equiv P_*(\emptyset_x, d)} \mathbf{y}dx$$

Proof of the Yoneda Lemma IV

Proof (cont'd).

Given $x, x' : B$, for any arrow $u : \hom_B(x, x')$, consider the naturality square induced by the action of χ on u:

$$
\begin{tikzcd}
d \arrow[r, equal] \arrow[d, "\chi_x"] \arrow[dr, "\chi_{x'}"] & d \arrow[d, "\chi_{x'}"] \\
\mathbf{y}(d, x) \arrow[r, "\mathbf{y}(d,u)"'] & \mathbf{y}(d, x')
\end{tikzcd}
$$

$$
\begin{tikzcd}
b \arrow[d, dashed, "\emptyset_x"'] \arrow[dr, dashed, "\emptyset_{x'}"] & \\
x \arrow[r, "u"'] & x'
\end{tikzcd}
$$

By right cancelation, $\mathbf{y}(d, u)$ must be a cocartesian arrow. $\square$

Proof of the Yoneda Lemma V

We can thus restrict to:

$$\begin{tikzcd} \prod\limits_{B}^{\mathrm{cocart}} P \arrow[r, bend right, "\mathrm{ev}_b"'] & P\,b \arrow[l, bend right, "\mathbf{y}"'] \end{tikzcd}$$

Step 3:

Proposition ([RV21], Thm 5.7.13; discrete case: [RS17], Thm. 9.7)

Let B be a Rezk type, $b : B$ an initial object, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at b given by

$$\mathrm{ev}_b : T :\equiv \prod_{B}^{\mathrm{cocart}} P \to P\,b$$

is an equivalence.

Proof of the Yoneda Lemma VI

Proof.

We do a round trip using $\mathbf{y}$. One direction is clear since cocartesian lifts of identities are themselves identites: $\mathrm{ev}_b(\mathbf{y}d) = \mathbf{y}d(b) = (\mathrm{id}_b)_* d = d$.
For the other direction, we want to define a natural transformation

$$\varepsilon : \big(\mathbf{y} \circ \mathrm{ev}_b \Rightarrow \mathrm{id}_T\big) \simeq \prod_{\substack{\sigma : T \\ x : B}} \big(\mathbf{y}(\sigma b)(x) \to \sigma(x)\big).$$

Let $\varepsilon_{\sigma,x}$ be the following filler:

$$\begin{tikzcd}
 & & \sigma(x) \\
\sigma(b) \arrow[urr, "\sigma(\emptyset_x)"] \arrow[rr, "\#" description, "P_*(\emptyset_x, \sigma(b))"'] & & \mathbf{y}(\sigma(b), x) \arrow[u, dashed, "\varepsilon_{\sigma,x}"'] \\
b \arrow[rr, "\emptyset_x"] & & x
\end{tikzcd}$$

Proof of the Yoneda Lemma VII

Proof (cont'd).

Now σ is cocartesian by assumption:

$$
\begin{tikzcd}
& \sigma(x) \\
\sigma(b) \arrow[ur, "\sigma(\emptyset_x)"] \arrow[r, "P_*(\emptyset_x{,}\sigma(b))"'] & \mathbf{y}(\sigma(b), x) \arrow[u, dashed, "\varepsilon_{\sigma,x}"'] \\
b \arrow[r, "\emptyset_x"] & x
\end{tikzcd}
$$

By right cancelation, $\varepsilon_{\sigma,x}$ must be cocartesian as well.

Proof of the Yoneda Lemma VIII

Proof (cont'd).

Now σ is cocartesian by assumption:

$$\begin{tikzcd}
 & \sigma(x) \\
\sigma(b) \arrow[ur, "\sigma(\emptyset_x)"] \arrow[r, "P_*(\emptyset_x{,}\sigma(b))"'] & \mathbf{y}(\sigma(b), x) \arrow[u, "\varepsilon_{\sigma,x}"'] \\
b \arrow[r, "\emptyset_x"] & x
\end{tikzcd}$$

By right cancelation, $\varepsilon_{\sigma,x}$ must be cocartesian as well.

Proof of the Yoneda Lemma IX

Proof (cont'd).

Now σ is cocartesian by assumption:

$$
\begin{tikzcd}
 & \sigma(x) \\
\sigma(b) \arrow[ur, "\sigma(\emptyset_x)"] \arrow[r, "P_*(\emptyset_x{,}\sigma(b))"'] & \mathbf{y}(\sigma(b),x) \arrow[u, "\varepsilon_{\sigma,x}"', "\cong"] \\
b \arrow[r, "\emptyset_x"] & x
\end{tikzcd}
$$

By right cancelation, $\varepsilon_{\sigma,x}$ must be cocartesian as well. But then it is a cocartesian arrow over an identity, hence itself an identity. $\square$

Proof of the Yoneda Lemma X

Step 4 (final): First note:

Lemma ([RS17], Lem. 9.8)

Let B be a Segal type. For any term $b : B$, the identity morphism $\mathrm{id}_b : b \downarrow B$ is an initial object.

Proof.

$$\begin{tikzcd} & b \arrow[dr, "u"] & \\ b \arrow[ur, equal, "\mathrm{id_b}"] \arrow[rr, dotted, "u"'] & & x \end{tikzcd}$$

$\square$

Proof of the Yoneda Lemma XI

As corollaries:

Theorem (Dependent Yoneda Lemma and Yoneda Lemma ([RV21], Thm. 5.7.2 & Thm. 5.7.3))

1. ***Dependent Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $Q : b \downarrow B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b is an equivalence:*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} Q \xrightarrow{\simeq} Q(\mathrm{id}_b)$$

2. ***Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b as in*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} \partial_1^* P \xrightarrow{\simeq} P\, b$$

is an equivalence, where $\partial_1 : b \downarrow B \to B$.

Outline

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Thank you!