Allegories and Bisimulations

A category-theoretic take on relations

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Outline

- Section 0: Allegories
- Section 1: Back-and-Forth
- Section 2: Modal Logic (time-permitting)

Some sources

- *Categories, Allegories* (Freyd-Scedrov)
- *Sketches of an Elephant* (Johnstone)
- https://ncatlab.org/nlab/show/allegory
- Wikipedia has a good article: https://en.wikipedia.org/wiki/Allegory_(mathematics)

0 Background: the Allegory of Relations

Rel

Rel is a standard example of a category:

Defn. **Rel** is the category whose

- objects are sets
- morphisms are binary relations:

$$\hom_{\mathbf{Rel}}(A, B) = \{R \mid R \subseteq A \times B\} = \mathcal{P}(A \times B)$$

- composition operation is given by

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \ (a, b) \in R \ \& \ (b, c) \in S\}$$

for $R \in \hom_{\mathbf{Rel}}(A, B), S \in \hom_{\mathbf{Rel}}(B, C)$.

Set versus **Rel**

It is customary to regard **Set** as a subcategory of **Rel**: the inclusion functor takes a function $f : A \to B$ to its *graph* $\{(a, f(a)) \mid a \in A\} \subseteq A \times B$, which we'll also call f:

$$f \in \hom_{\mathbf{Rel}}(A, B)$$

Set

- has all small limits and colimits
- has exponentials and a subobject classifier
- has cartesian closed slice categories
- ...

Rel

- doesn't (for the most part)

Pos-enrichment

Notice:

$$\mathrm{hom}_{\mathbf{Rel}}(A, B) = \mathcal{P}(A \times B)$$

- $\subseteq$ is a partial order on $\mathrm{hom}_{\mathbf{Rel}}(A, B)$:
  - For all $R \in \mathrm{hom}_{\mathbf{Rel}}(A, B)$, $R \subseteq R$
  - For all $R, R', R'' \in \mathrm{hom}_{\mathbf{Rel}}(A, B)$, if $R \subseteq R'$ and $R' \subseteq R''$, then $R \subseteq R''$
  - If $R \subseteq R'$ and $R' \subseteq R$, then $R = R'$
- $\subseteq$ is compatible with composition: for $R : A \to B$, $S, S' : B \to C$ and $T : C \to D$ in **Rel**,

$$S \subseteq S' \implies (S \circ R) \subseteq (S' \circ R)$$
$$S \subseteq S' \implies (T \circ S) \subseteq (T \circ S')$$

Pos-valued representables

The previous slide can be summarized as asserting that the following are functors for any set C:

$$
\begin{aligned}
\hom_{\mathbf{Rel}}(-, C) &: \mathbf{Rel}^{\mathrm{op}} \to \mathbf{Pos} \\
&: A \qquad\qquad \mapsto (\hom_{\mathbf{Rel}}(A, C), \subseteq) \\
&: R \subseteq A \times B \;\; \mapsto (- \circ R) : (\hom_{\mathbf{Rel}}(B, C), \subseteq) \to (\hom_{\mathbf{Rel}}(A, C), \subseteq) \\
\hom_{\mathbf{Rel}}(C, -) &: \mathbf{Rel} \to \mathbf{Pos} \\
&: D \qquad\qquad \mapsto (\hom_{\mathbf{Rel}}(C, D), \subseteq) \\
&: U \subseteq D \times E \;\; \mapsto (U \circ -) : (\hom_{\mathbf{Rel}}(C, D), \subseteq) \to (\hom_{\mathbf{Rel}}(C, E), \subseteq)
\end{aligned}
$$

We have the meets

Given $R, R' \in \hom_{\mathbf{Rel}}(A, B)$,

$$R \cap R' \in \hom_{\mathbf{Rel}}(A, B)$$

This satisfies some nice properties, e.g.

- $S \circ (R \cap R') \subseteq (S \circ R) \cap (S \circ R')$
- $(S \cap S') \circ R \subseteq (S \circ R) \cap (S' \circ R)$
- $R \cap R = R, \quad R \cap R' = R' \cap R, \quad R \cap (R' \cap R'') = (R \cap R') \cap R''$

Also: nullary intersections ($A \times B \in \hom_{\mathbf{Rel}}(A, B)$), infinitary intersections, binary and infinitary unions, nullary unions (the empty relation), etc.

Dagger Categories

For any $R \in \mathrm{hom}_{\mathbf{Rel}}(A, B)$,

$$R^\dagger := \{(b, a) \mid (a, b) \in R\} \in \mathrm{hom}_{\mathbf{Rel}}(B, A)$$

Defn. A **dagger category** $\mathbb{C}$ is a category equipped with a contravariant endofunctor $\dagger : \mathbb{C}^{\mathrm{op}} \to \mathbb{C}$ such that

- $\dagger$ is the identity on objects
- $\dagger$ is an **involution**: $\dagger \circ \dagger = \mathrm{id}_{\mathbb{C}}$

This structure doesn't exist (or is trivial) for **Set**

For any functions $f, f' : A \to B$,

- $f \subseteq f'$ if and only if $f = f'$
- $f \cap f'$ is only a function if $f = f'$ (in which case $f \cap f' = f = f'$)
- $f^\dagger$ is only a function if f is a bijection

Allegories

Defn. An **allegory** $\mathbb{C}$ is a category equipped with

- A poset structure $\leq$ on each hom-set which is compatible with composition (i.e. the function $(- \circ -) : \hom_{\mathbb{C}}(B, C) \times \hom_{\mathbb{C}}(A, B) \to \hom_{\mathbb{C}}(A, C)$ is monotone)and has binary meets (for any $R, R' \in \hom_{\mathbb{C}}(A, B)$, $R \wedge R'$ is the greatest lower bound of R and R')
- An involution $\dagger : \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$

such that

- Involution distributes over meets: $(R \wedge R')^{\dagger} = R^{\dagger} \wedge R'^{\dagger}$
- Composition semi-distributes over meets:

$$S \circ (R \wedge R') \leq (S \circ R) \wedge (S \circ R')$$
$$(S \wedge S') \circ R \leq (S \circ R) \wedge (S' \circ R)$$

- The **modular law** is satisfied:

$$(S \circ R) \wedge T \leq (S \wedge (T \circ R^{\dagger})) \circ R$$

Rel is an allegory

For any $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq A \times C$:

$$
\begin{aligned}
(a,c) \in (S \circ R) \cap T &\iff (a,c) \in S \circ R \text{ and } (a,c) \in T \\
&\iff \exists b \in B\ (a,b) \in R \text{ and } (b,c) \in S \text{ and } (a,c) \in T
\end{aligned}
$$

$$
\begin{aligned}
(a,b) \in R \text{ and } (a,c) \in T &\iff (b,a) \in R^{\dagger} \text{ and } (a,c) \in T \\
&\implies (b,c) \in T \circ R^{\dagger}
\end{aligned}
$$

$$
\begin{aligned}
&(b,c) \in S \text{ and } (b,c) \in T \circ R^{\dagger} \text{ and } (a,b) \in R \\
\iff &(b,c) \in S \cap (T \circ R^{\dagger}) \text{ and } (a,b) \in R \\
\implies &(a,c) \in (S \cap (T \circ R^{\dagger})) \circ R
\end{aligned}
$$

Simple and Entire morphisms

Note: if $R \in \hom_{\mathbb{C}}(A, B)$ for some allegory $\mathbb{C}$,

$$R \circ R^\dagger \in \hom_{\mathbb{C}}(B, B) \qquad \text{and} \qquad R^\dagger \circ R \in \hom_{\mathbb{C}}(A, A)$$

Defn. A morphism $R : A \to B$ in some allegory is said to be

- **simple** if $R \circ R^\dagger \leq \mathrm{id}_B$,
- **cosimple** if $R^\dagger \circ R \leq \mathrm{id}_A$,
- **entire** if $\mathrm{id}_A \leq R^\dagger \circ R$, and
- **coentire** if $\mathrm{id}_B \leq R \circ R^\dagger$.

The (co)simple and (co)entire morphisms of **Rel**

Prop. A relation $R \in \mathrm{hom}_{\mathbf{Rel}}(A, B)$ is

- simple iff it is a partial function: $(a, b) \in R$ and $(a, b') \in R$ implies $b = b'$
- cosimple iff it is injective: $(a, b) \in R$ and $(a', b) \in R$ implies $a = a'$
- entire iff it is entire (or *total*): for all $a \in A$, there exists $b \in B$ such that $(a, b) \in R$
- coentire iff it is surjective: for all $b \in B$, there exists $a \in A$ such that $(a, b) \in R$

Note **Set** is the subcategory of simple, entire relations

Some general results (for an arbitrary allegory)

Prop. All isomorphisms are simple, cosimple, entire, and coentire

Prop. The class of simple morphisms, the class of cosimple morphisms, the class of entire morphisms, and the class of coentire morphisms are closed under composition

Prop. The class of (co)simple morphisms is downward closed:

$$R \leq R' \text{ and } R' \text{ is (co)simple} \qquad \Longrightarrow \qquad R \text{ is (co)simple}$$

Prop. The class of (co)entire morphisms is upward closed:

$$R \leq R' \text{ and } R \text{ is (co)entire} \qquad \Longrightarrow \qquad R' \text{ is (co)entire}$$

Prop. R is entire iff $R^\dagger$ is coentire (and similarly for (co)simple)

The allegory of internal relations of a regular category

We can "internalize" the notion of a relation: given a category $\mathbb{C}$ with binary products, a subobject

$$R \rightarrowtail A \times B$$

is an **internal binary relation** between A and B.

We can form the category $\mathbf{Rel}(\mathbb{C})$ with the same objects as $\mathbb{C}$, and whose morphisms $A \to B$ are internal binary relations between A and B.

Thm. If $\mathbb{C}$ is a **regular** category, then $\mathbf{Rel}(\mathbb{C})$ is an allegory

Prop. $\mathbf{Set}$ is a regular category, and $\mathbf{Rel} = \mathbf{Rel}(\mathbf{Set})$

1 Allegories with Back-and-Forth Classes

Top

Defn. A **topology** on a set X is a collection $\tau \subseteq \mathcal{P}(X)$ (the elements of τ are called **open subsets** of X) such that

- $\emptyset, X \in \tau$
- If $U, U' \in \tau$, then $U \cap U' \in \tau$
- If I is a set and $U_i \in \tau$ for each $i \in I$,

$$\left(\bigcup_{i \in I} U_i\right) \in \tau.$$

Defn. **Top** is the category whose

- objects are topological spaces: pairs (X, τ) where τ is a topology on X
- morphisms are **continuous functions**: $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if

$$U \in \tau_Y \qquad \Longrightarrow \qquad f^{-1}(U) \in \tau_X$$

Continuous Relations don't give rise to an allegory

We can generalize this to relations:

Defn. Given topological spaces (A, τ_A) and (B, τ_B) and $R \subseteq A \times B$, R is said to be **continuous** if

$$U \in \tau_B \qquad \Longrightarrow \qquad R^{-1\dagger}(U) \in \tau_A$$

where $R^\dagger(U) = \{a \in A \mid (u, a) \in R^\dagger \text{ for some } u \in U\}$.

Problem: R continuous does not imply $R^\dagger$ continuous, so the category of continuous relations (which *is* a category), is *not* an allegory.

Proposed approach: **TopRel**

Defn. **TopRel** is the category whose

- objects are topological spaces (A, τ_A)
- morphisms $R : (A, \tau_A) \to (B, \tau_B)$ are binary relations $R \subseteq A \times B$. (no assumptions on R – ignore the topologies for now)

Prop. **TopRel** is an allegory.

Proof is identical to the proof that **Rel** is an allegory

Defn. Write **Back** for the class of continuous morphisms in **TopRel**

Defn. Write **Forth** for the class of **open** morphisms in **TopRel**:
morphisms $R \in \hom_{\mathbf{TopRel}}((A, \tau_A), (B, \tau_B))$ such that

$$U \in \tau_A \qquad \implies \qquad R(U) \in \tau_B.$$

Some basic results

- $R \in$ **Forth** if and only if $R^\dagger \in$ **Back**
- Both **Back** and **Forth** are closed under composition and contain all **Top**-isos (homeomorphisms)
- **Top** is the subcategory of continuous, entire, simple **TopRel**-morphisms
- Quotient maps $X \to X/\sim$ in **Top** are open and coentire, but are only cosimple if $\sim$ is identity
- A **TopRel**-iso (a bijection) is a **Top**-iso (a homeomorphism) iff it is in **Forth** and **Back**

Another example: DynRel

Defn. A **dynamic set** is a pair (A, f) where A is a set and $f : A \rightharpoonup A$ is a partial function (a simple **Rel**-endomorphism).

Defn. **DynRel** is the allegory of dynamic sets and binary relations, whose back and forth classes are given by:

- $R : (A, f) \to (B, g)$ is in **Forth** if

$$f(a) \text{ is defined and } (a, b) \in R \quad \Longrightarrow \quad g(b) \text{ is defined and } (f(a), g(b)) \in R$$

- $R : (A, f) \to (B, g)$ is in **Back** if

$$g(b) \text{ is defined and } (a, b) \in R \quad \Longrightarrow \quad f(a) \text{ is defined and } (f(a), g(b)) \in R$$

Defn. For a set Σ, a **Σ-dynamic set** is a set A equipped with a Σ-indexed family of partial functions $\{f_\sigma : A \rightharpoonup A\}_{\sigma\in\Sigma}$.
Defn. **Σ-DynRel** is the allegory of Σ-dynamic sets and binary relations.
For each $\sigma \in \Sigma$, there are classes of morphisms, σ-**Forth** and σ-**Back**, defined as above.

So, for each binary relation R, there is some subset $\Pi \subseteq \Sigma$ of all those σ such that R is in σ-**Forth** (or σ-**Back**, or both).

2 Modal Logics and Bisimulation

Valuations

The model theory of *classical logic* makes use of **valuations**: functions which "assign truth values" to atomic propositions

$$v : \Phi \to \{0, 1\}$$

We can generalize this somewhat:

$$v : \Phi \to \mathcal{P}(X)$$

$v(p)$ is the *extension* of p, or the set of "states where p is true".

Dynamic Modal Logic

We can pair together a dynamic set (A, f) with a valuation $v : \Phi \to \mathcal{P}(A)$ to get a **dynamic model**.

Dynamic models *interpret* the language $\mathcal{L}_{\bigcirc}$:

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \bigcirc\varphi \qquad (p \in \Phi)$$

We define a function $[\![-]\!] : \mathcal{L}_{\bigcirc} \to \mathcal{P}(A)$ recursively by

$$\begin{aligned} [\![p]\!] &= v(p) && (p \in \Phi) \\ [\![\neg\varphi]\!] &= A \setminus [\![\varphi]\!] \\ [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\bigcirc\varphi]\!] &= f^{-1}[\![\varphi]\!] \end{aligned}$$

Defn. A **bisimulation** between dynamic models (A, f, v_A) and (B, g, v_B) is a binary relation $S \in \hom_{\mathbf{DynRel}}((A, f), (B, g))$ in both the **Forth** and **Back** classes, which also satisfies the **Base** condition: for any $(a, b) \in S$ and $p \in \Phi$,

$$a \in v_A(p) \qquad \iff \qquad b \in v_B(p).$$

Bisimulation Invariance

Thm. For dynamic models (A, f, v_A) and (B, g, v_B) and a bisimulation S between them,

- If $(a, b) \in S$, then for any $\varphi \in \mathcal{L}_{\bigcirc}$,
$$a \in [\![\varphi]\!]_A \qquad \Longleftrightarrow \qquad b \in [\![\varphi]\!]_B$$
- If S is **Entire**, then
$$[\![\varphi]\!]_B = B \qquad \Longrightarrow \qquad [\![\varphi]\!]_A = A$$
- If S is **Coentire**, then
$$[\![\varphi]\!]_A = A \qquad \Longrightarrow \qquad [\![\varphi]\!]_B = B$$

Multiple dynamics

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \bigcirc_\sigma \varphi \qquad (p \in \Phi,\ \sigma \in \Sigma)$$

$$[\![\bigcirc_\sigma \varphi]\!] = f_\sigma^{-1}[\![\varphi]\!]$$

For $\Pi \subseteq \Sigma$, a Π-bisimulation between Σ-dynamic models $(A, \{f_\sigma\}_{\sigma\in\Sigma}, v_A)$ and $(B, \{g_\sigma\}_{\sigma\in\Sigma}, v_B)$ is a relation satisfying **Base**, and π-**Forth** and π-**Back** for each $\pi \in \Pi$.

Topological Modal Logic

We can instead use a topological structure to interpret $\Box$. Define $\mathcal{L}_\Box$ by

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \qquad (p \in \Phi)$$

A **topological model** (A, τ_A, v) interprets $\mathcal{L}_\Box$:

$$\begin{aligned} [\![p]\!] &= v(p) && (p \in \Phi) \\ [\![\neg\varphi]\!] &= A \setminus [\![\varphi]\!] \\ [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\Box\varphi]\!] &= \mathrm{int}([\![\varphi]\!]) \end{aligned}$$

where int denotes topological interior (with respect to τ_A).

Topo-bisimulations

Defn. A **topo-bisimulation** between topological models (A, τ_A, v_A) and (B, τ_B, v_B) is a **TopRel**-morphism in **Forth** and **Back** (open & continuous) that satisfies **Base**.

Thm. For topological models (A, τ_A, v_A) and (B, τ_B, v_B) and a bisimulation S between them,

- If $(a, b) \in S$, then for any $\varphi \in \mathcal{L}_\Box$,
$$a \in [\![\varphi]\!]_A \iff b \in [\![\varphi]\!]_B$$
- If S is **Entire**, then
$$[\![\varphi]\!]_B = B \implies [\![\varphi]\!]_A = A$$
- If S is **Coentire**, then
$$[\![\varphi]\!]_A = A \implies [\![\varphi]\!]_B = B$$

Dynamic Topological Modal Logic

My research (in particular my master's thesis) explores bisimulations of **dynamic topological models**, which are models $(A, \tau_A, \{f_\sigma\}_{\sigma\in\Sigma}, v_A)$ interpreting

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \bigcirc_\sigma \varphi$$

with the appropriate notion of bisimulation.

This has an interesting philosophical interpretation if we read $\Box\varphi$ as "φ is knowably (or verifiably) true" and $\bigcirc_\sigma\varphi$ as "after performing (or executing) σ, φ holds".

Thank you!