A Higher Inductive Presentation of the Integers

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Common Definitions of the Integers

1st Definition

$\mathbb{Z}$ is the inductive type generated by

- $0_{\mathbb{Z}} : \mathbb{Z}$,
- $\mathsf{pos} : \mathbb{N} \to \mathbb{Z}$,
- $\mathsf{neg} : \mathbb{N} \to \mathbb{Z}$.

In other words, $\mathbb{Z} \simeq \mathbb{N} + \mathbf{1} + \mathbb{N}$.

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2nd Definition

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Problem: can only eliminate into sets.

Common Definitions of the Integers

Recall: the integers are initial in the category of pointed sets with an automorphism.

3^{rd} Definition

Let $\mathbb{Z}_a$ be the higher inductive type generated by

- $0_{\mathbb{Z}_a} : \mathbb{Z}_a$
- $\mathsf{succ}_{\mathbb{Z}_a} : \mathbb{Z}_a \to \mathbb{Z}_a$,
- $\mathsf{pred}_{\mathbb{Z}_a} : \mathbb{Z}_a \to \mathbb{Z}_a$,
- $\mathsf{sec}_{\mathbb{Z}_a} : \mathsf{pred}_{\mathbb{Z}_a} \circ \mathsf{succ}_{\mathbb{Z}_a} \sim id$,
- $\mathsf{ret}_{\mathbb{Z}_a} : \mathsf{succ}_{\mathbb{Z}_a} \circ \mathsf{pred}_{\mathbb{Z}_a} \sim id$,
- $\mathsf{isSet}(\mathbb{Z}_a)$

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Common Definitions of the Integers

Paolo Capriotti: integers as a pointed type with an autoequivalence.

4^{th} Definition

Let $\mathbb{Z}_h$ be the higher inductive type generated by:

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- $\mathsf{coh}_{\mathbb{Z}_h} : \mathsf{succ}_{\mathbb{Z}_h} \circ \mathsf{sec}_{\mathbb{Z}_h} \sim \mathsf{ret}_{\mathbb{Z}_h} \circ \mathsf{succ}_{\mathbb{Z}_h}$.

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Problem: Does this HIT really represent the integers?

Integers as HIT

Altenkirch & Scoccola [1] tried to prove that $\mathbb{Z} \simeq \mathbb{Z}_h$.
Problem: path algebra is too complicated due to 2-path constructor $\mathsf{coh}_{\mathbb{Z}_h}$.

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Solutions:

1. Cavallo [3], Altenkirch & Scoccolla [1]: Use bi-invertible maps rather than half-adjoint equivalences.

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Solutions:

1. Cavallo [3], Altenkirch & Scoccolla [1]: Use bi-invertible maps rather than half-adjoint equivalences.
2. Use a different induction principle.

An Induction Principle for $\mathbb{Z}_h$

Dependent UMP of $\mathbb{Z}$ (Rijke [4])

Consider a type family $E : \mathbb{Z} \to \mathcal{U}$ equipped with a point $e_0 : E(0_{\mathbb{Z}})$ and a family of bi-invertible maps

$$s_E : \prod_{z:\mathbb{Z}} E(z) \simeq E(\mathrm{succ}_{\mathbb{Z}}(z))$$

Then, there is a dependent function $f : \prod_{z:\mathbb{Z}} E(z)$ such that

- $f(0_{\mathbb{Z}}) = e_0$ and,
- for every $z : \mathbb{Z}$, $f(\mathrm{succ}_{\mathbb{Z}}(z)) = s_E(z, f(z))$

Idea: use this induction principle with half-adjoint equivalences.

What Justifies an Induction Principle?

Given a suitable notion of category, the induction principle should be

- weak enough to follow from initiality, and
- strong enough to pin the HIT down.

In other words, we want **inductivity to be equivalent to initiality**.

General strategy: Awodey, Gambino & Sojakova [2], Sojakova [5] [6].

Naïve Categories

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$$f : \mathrm{Hom}_{\mathcal{C}}(A, B), g : \mathrm{Hom}_{\mathcal{C}}(B, C) \vdash g \circ f : \mathrm{Hom}_{\mathcal{C}}(A, C)$$

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- identity morphisms, $1_A : \mathrm{Hom}_{\mathcal{C}}(A, A)$,
- for each $f : \mathrm{Hom}_{\mathcal{C}}(A, B)$, "unitors"

$$\rho_f : f \circ 1_A = f \text{ and } \lambda_f : 1_B \circ f = f,$$

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- for each $f : \mathrm{Hom}_{\mathcal{C}}(A, B)$, "unitors"

$$\rho_f : f \circ 1_A = f \text{ and } \lambda_f : 1_B \circ f = f,$$

- for each composable triple of morphisms f, g, h, an "associator"

Homotopy Initiality

Let A be an object in a naïve category $\mathcal{C}$.

Definition

Following [2], we say that A is **homotopy initial** in $\mathcal{C}$ when the following type is inhabited:

$$\mathsf{ishinit}(A) :\equiv \prod_{B:\mathcal{C}_0} \mathsf{isContr}(\mathsf{Hom}_{\mathcal{C}}(A, B)).$$

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Lemma

The previous type is equivalent to the product of

$$\mathsf{hasrec}(A) :\equiv \prod_{B:\mathcal{C}_0} \mathsf{Hom}_{\mathcal{C}}(A, B)$$

and

$$\mathsf{hasrecunique}(A) :\equiv \prod_{B:\mathcal{C}_0} \mathsf{isProp}(\mathsf{Hom}_{\mathcal{C}}(A, B))$$

The Naïve Category of $\mathbb{Z}$-Algebras

Definition

A $\mathbb{Z}$-**algebra** (in universe $\mathcal{U}$) is a pointed type with a half-adjoint autoequivalence.

$$\mathbb{Z}\mathrm{Alg} :\equiv \sum_{A:\mathcal{U}} A \times (A \simeq A).$$

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Example

There is a canonical $\mathbb{Z}$-algebra structure on $\mathbb{Z}$ given by $0_{\mathbb{Z}}$ and the autoequivalence defined by:

- $\mathsf{succ}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$,
- $\mathsf{pred}_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$,
- $\mathsf{sec}_{\mathbb{Z}} : \mathsf{pred}_{\mathbb{Z}} \circ \mathsf{succ}_{\mathbb{Z}} \sim id$,
- $\mathsf{ret}_{\mathbb{Z}} : \mathsf{succ}_{\mathbb{Z}} \circ \mathsf{pred}_{\mathbb{Z}} \sim id$,
- $\mathsf{coh}_{\mathbb{Z}} : \mathsf{succ}_{\mathbb{Z}} \circ \mathsf{sec}_{\mathbb{Z}} \sim \mathsf{ret}_{\mathbb{Z}} \circ \mathsf{succ}_{\mathbb{Z}}$.

The Naïve Category of $\mathbb{Z}$-Algebras

Notation

Given a $\mathbb{Z}$-algebra $\mathbf{A}$, we shall typically denote its components as follows:

- Underlying type:
  - $A : \mathcal{U}$
- Point:
  - $a_0 : A$
- Autoequivalence:
  - $s_A : A \to A$
  - $p_A : A \to A$
  - $\sigma_A : p_A \circ s_A \sim id_A$
  - $\sigma_A : s_A \circ p_A \sim id_A$
  - $\tau_A : s_A \circ \sigma_A \sim \rho_A \circ s_A$

The Naïve Category of $\mathbb{Z}$-Algebras

Definition

A $\mathbb{Z}$-**algebra morphism** from $\mathbf{A} : \mathbb{Z}\mathrm{Alg}$ to $\mathbf{B} : \mathbb{Z}\mathrm{Alg}$ is a pointed, equivalence-preserving map.

$$\mathrm{Hom}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{A}, \mathbf{B}) :\equiv \sum_{f : A \to B} (f(a_0) = b_0) \times (f \circ s_A \sim s_B \circ f).$$

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Example (Identity Morphism)

For any $\mathbf{A} : \mathbb{Z}\mathrm{Alg}$, there is a morphism $1_A : \mathrm{Hom}_{\mathbb{Z}}\mathrm{Alg}(\mathbf{A}, \mathbf{A})$ given by

- $id_A : A \to A$
- $refl_{a_0} : a_0 = a_0$
- $\lambda a.\ refl_{s_A(a)} : id_A \circ s_A \sim s_A \circ id_A$

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The Naïve Category of $\mathbb{Z}$-Algebras

Definition (Composition)

Let $\mathbf{f} : \mathrm{Hom}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{A}, \mathbf{B})$ and $\mathbf{g} : \mathrm{Hom}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{B}, \mathbf{C})$. Their composite $\mathbf{g} \circ \mathbf{f} : \mathrm{Hom}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{A}, \mathbf{C})$ is defined as the following triple:

- $g \circ f : A \to C$
- $g(f_0) \cdot g_0 : g(f(a_0)) = c_0$
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Lemma

1. *Composition of $\mathbb{Z}$-algebras is associative*
2. *Identity $\mathbb{Z}$-algebra morphisms satisfy left and right unit laws with respect to composition.*

Fibered $\mathbb{Z}$-Algebras

Recall the dependent UMP of $\mathbb{Z}$:

Consider a type family $E : \mathbb{Z} \to \mathcal{U}$ equipped with a point $e_0 : E(0_{\mathbb{Z}})$ and a family of equivalences

$$s_E : \prod_{z:\mathbb{Z}} E(z) \simeq E(\mathrm{succ}_{\mathbb{Z}}(z))$$

Then, there is a dependent function $f : \prod_{z:\mathbb{Z}} E(z)$ such that

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Fibered $\mathbb{Z}$-Algebras

Definition

A **fibered $\mathbb{Z}$-algebra** over $\mathbf{A}$ is a type family $E : A \to \mathcal{U}$ together with

- a point $e_0 : E(a_0)$ over a_0, and
- a fiberwise equivalence $s_E : \prod_{a:A} E(a) \to E(s_A(a))$ over s_A.

We denote the type of all such algebras by $\mathrm{Fib}\mathbb{Z}\mathrm{Alg}(\mathbf{A})$.

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Example

Every $\mathbb{Z}$-algebra $\mathbf{B} :\equiv (B, b_0, (s_B, i_B))$ induces a constant fibered $\mathbb{Z}$-algebra over $\mathbf{A}$ given by

- $E :\equiv \lambda a.\, B : A \to \mathcal{U}$,
- $e_0 :\equiv b_0 : B$,
- $s_E :\equiv \lambda a.\, s_B : \prod_{a:A} B \to B$,
- $i_E :\equiv \lambda a.\, i_B : \prod_{a:A} \mathrm{isequiv}(s_B)$,

Fibered $\mathbb{Z}$-Algebra Sections

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Fibered $\mathbb{Z}$-Algebra Sections

Definition

A **section** of a fibered $\mathbb{Z}$-algebra $\mathbf{E}$ over $\mathbf{A}$ is a pointed, equivalence-preserving dependent function $f : \prod_{a:A} E(a)$.

$$\mathrm{Sec}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{A}, \mathbf{E}) :\equiv \sum_{f:\prod_{a:A} E(a)} (f(a_0) = e_0) \times \left(\prod_{a:A} f(s_A(a)) = s_E(a, f(a)) \right)$$

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Remark

If $\mathbf{E}$ is the constant fibered $\mathbb{Z}$-algebra over $\mathbf{A}$ induced $\mathbb{Z}$-algebra $\mathbf{B}$, then

$$\mathrm{Sec}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{A}, \mathbf{E}) \equiv \mathrm{Hom}_{\mathbb{Z}\mathrm{Alg}}(\mathbf{A}, \mathbf{B})$$

Inductive $\mathbb{Z}$-Algebras

Definition

A is **inductive** if every fibered $\mathbb{Z}$-algebra over it has a section.

$$\mathsf{isind}(\mathbf{A}) :\equiv \prod_{\mathbf{E} : \mathsf{Fib}\mathbb{Z}\mathsf{Alg}(\mathbf{A})} \mathsf{Sec}_{\mathbb{Z}\mathsf{Alg}}(\mathbf{A}, \mathbf{E})$$

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Example

The higher inductive type $\mathbb{Z}_h$ is, by definition, an inductive $\mathbb{Z}$-algebra.

Roadmap

Our ultimate goal is:

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It suffices to show:

1. For every $\mathbf{A} : \mathbb{Z}\mathsf{Alg}$, $\mathsf{isind}(\mathbf{A}) \simeq \mathsf{ishinit}(\mathbf{A})$.
2. $\mathsf{ishinit}(\mathbb{Z})$.
3. There is a unique homotopy initial $\mathbb{Z}$-algebra.

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For (1), in turn, it suffices to show:

3. For every $\mathbf{A} : \mathbb{Z}\mathsf{Alg}$, $\mathsf{isind}(\mathbf{A}) \leftrightarrow \mathsf{ishinit}(\mathbf{A})$,
4. For every $\mathbf{A} : \mathbb{Z}\mathsf{Alg}$, $\mathsf{isind}(\mathbf{A})$ and $\mathsf{ishinit}(\mathbf{A})$ are propositions.

Identity Type of Sections

The previous claims will require comparisons of sections or morphisms, so we begin by characterizing their identity types "extensionally."

Lemma

Consider any fibered $\mathbb{Z}$-algebra $\mathbf{E}$ *over* $\mathbf{A}$. *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be two sections of* $\mathbf{E}$. *Then,*

$$(\mathbf{f} = \mathbf{g}) \simeq \mathrm{SecId}(\mathbf{f}, \mathbf{g}),$$

where $\mathrm{SecId}(\mathbf{f}, \mathbf{g})$ *is the type of triples* (H, H_0, H_s) *such that*

- $H : f \sim g$
- $H_0 : H(a_0) = f_0 \cdot g_0^{-1}$
- $H_s : \prod_{a:A} H(s_A(a)) = f_s(a) \cdot s_E(a, H(a)) \cdot g_s(a)^{-1}$

Proof.

Fix $\mathbf{f} : \mathsf{Sec}_{\mathbb{Z}\mathsf{Alg}}(\mathbf{A}, \mathbf{E})$. The relation SecId is reflexive, so there is a map

$$\prod_{\mathbf{g}:\mathsf{Sec}_{\mathbb{Z}\mathsf{Alg}}(\mathbf{A},\mathbf{E})} \mathbf{f} = \mathbf{g} \to \mathsf{SecId}(\mathbf{f}, \mathbf{g}).$$

To prove that it is a fiberwise equivalence, it suffices to show that

$$\sum_{\mathbf{g}:\mathsf{Sec}_{\mathbb{Z}\mathsf{Alg}(\mathbf{A},\mathbf{E})}} \mathsf{SecId}(\mathbf{f}, \mathbf{g})$$

is contractible. Pairing the first components of the summands, we obtain the type

$$\sum \left(g : \prod_{a:A} E(a) \right), f \sim g,$$

which is contractible. The same thing happens with the other summands. $\square$

Identity Type of Morphisms

Corollary

Consider any two $\mathbb{Z}$-algebras $\mathbf{A}$ *and* $\mathbf{B}$*. Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be two morphisms from* $\mathbf{A}$ *to* $\mathbf{B}$*. Then,*

$$(\mathbf{f} = \mathbf{g}) \simeq \mathsf{HomId}(\mathbf{f}, \mathbf{g}),$$

where $\mathsf{HomId}(\mathbf{f}, \mathbf{g})$ *is the type of triples* (H, H_0, H_s) *such that*

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Uniqueness Principle for Inductive $\mathbb{Z}$-Algebras

Lemma

$$\mathsf{isind}(\mathbf{A}) \to \prod_{\mathbf{E}:\mathsf{Fib}\mathbb{Z}\mathsf{Alg}(\mathbf{A})} \mathsf{isProp}(\mathsf{Sec}_{\mathbb{Z}\mathsf{Alg}}(\mathbf{A}, \mathbf{E})).$$

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Proof.

Suppose $\mathbf{A}$ is inductive. Fix two arbitrary sections $\mathbf{f}$ and $\mathbf{g}$ of $\mathbf{E}$. $\mathsf{SecId}(\mathbf{f}, \mathbf{g})$ is precisely the type of sections of the following fibered $\mathbb{Z}$-algebra:

- underlying type family: $f \sim g$,
- point: $f_0 \cdot g_0^{-1}$,
- fiberwise equivalence: for each $a : A$,

$$\begin{array}{ccc} f(a) = g(a) & \to & f(s_A(a)) = g(s_A(a)) \\ q & \mapsto & f_s(a) \cdot s_E(a, q) \cdot g_s(a)^{-1}, \end{array}$$

so the conclusion follows by $\mathbf{A}$-induction. $\square$

Inductivity Is a Property

Theorem

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Proof.

Fix $\mathbf{A}$. We may assume that it is inductive.
Recall that

$$\mathsf{isind}(\mathbf{A}) \equiv \prod_{\mathbf{E}:\mathsf{Fib}\mathbb{Z}\mathsf{Alg}(\mathbf{A})} \mathsf{Sec}_{\mathbb{Z}\mathsf{Alg}}(\mathbf{A}, \mathbf{E}).$$

Since propositions are closed under $\prod$, the conclusion follows from the Uniqueness Principle. $\square$

Inductivity Implies Initiality

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Every inductive $\mathbb{Z}$-algebra is homotopy initial.

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Suppose **A** is inductive and fix an arbitrary $\mathbb{Z}$-algebra **B**.

1. Morphism into **A**: by **A**-induction into constant fibered $\mathbb{Z}$-algebra induced by **B**.
2. Uniqueness: by Uniqueness Principle for inductive $\mathbb{Z}$-algebras.

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Corollary

$\mathbb{Z}_h$ is homotopy initial.

Initiality Implies Inductivity

Theorem

Every homotopy initial $\mathbb{Z}$-algebra is inductive.

Initiality Implies Inductivity

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Every homotopy initial $\mathbb{Z}$-algebra is inductive.

We need the following lemma:

Lemma

Every $\mathbf{E} : \mathrm{Fib}\mathbb{Z}\mathrm{Alg}(\mathbf{A})$ *has an associated "total $\mathbb{Z}$-algebra"* $\tilde{\mathbf{E}}$ *given by*

- *underlying type:* $\sum_{a:A} E(a)$
- *point:* (a_0, e_0)
- *autoequivalence:* $(a, e) \mapsto (s_A(a), s_E(a, e))$

and a projection morphism $\pi_1 : \mathrm{Hom}_{\mathbb{Z}\mathrm{Alg}}(\tilde{\mathbf{E}}, \mathbf{A})$.

Initiality Implies Inductivity

Proof.

Suppose $\mathbf{A}$ is homotopy initial and consider an arbitrary $\mathbf{E} : \mathrm{Fib}\mathbb{Z}\mathrm{Alg}(\mathbf{A})$. Let $\tilde{\mathbf{E}}$ be its associated $\mathbb{Z}$-algebra.

1. Get a morphism $\mathbf{f} :\equiv (f, f_0, f_s)$ into $\tilde{\mathbf{E}}$ by $\mathbf{A}$-recursion. Notice that
$$pr_2 \circ f : \prod_{a:A} E(pr_1(f(a)))$$
2. Get a path $\pi_1 \circ \mathbf{f} = 1_A$ by the uniqueness principle of $\mathbf{A}$.
3. Transport $pr_2 \circ f$ along this path to obtain a section of $\mathbf{E}$.

$\square$

Initiality Implies Inductivity

Proof.

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Corollary

For every $\mathbf{A} : \mathbb{Z}\mathrm{Alg}$,

$$\mathrm{isind}(\mathbf{A}) \simeq \mathrm{ishinit}(\mathbf{A}).$$

Roadmap

Ultimate goal is:

Theorem

$$\mathbb{Z} =_{\mathbb{Z}\mathsf{Alg}} \mathbb{Z}_h.$$

It suffices to show:

1. For every $\mathbf{A} : \mathbb{Z}\mathsf{Alg}$, $\mathsf{isind}(\mathbf{A}) \simeq \mathsf{ishinit}(\mathbf{A})$. ✓
2. $\mathsf{ishinit}(\mathbb{Z})$.
3. There is a unique homotopy initial $\mathbb{Z}$-algebra.

Uniqueness of Homotopy Initial $\mathbb{Z}$-Algebras

Paths between $\mathbb{Z}$-algebras are "equivalence morphisms":

Lemma

For any two $\mathbb{Z}$-algebras $\mathbf{A} :\equiv (A, a_0, s_A)$ and $\mathbf{B} :\equiv (B, b_0, s_B)$, it is the case that

$$(\mathbf{A} = \mathbf{B}) \simeq \left(\sum_{e : A \simeq B} (e(a_0) = b_0) \times (e \circ s_A = s_B \circ e) \right),$$

where we have identified e with its underlying map.

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Proof.

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Corollary

$$\mathrm{isProp}\left(\sum_{\mathbf{A}:\mathbb{Z}\mathrm{Alg}} \mathrm{ishinit}(\mathbf{A}) \right).$$

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Propositional Computation Rules for Other Constructors

- The alternative induction principle for $\mathbf{Z}_h$ only specifies the behavior of the dependent eliminator on 0_h and succ_h.

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- The alternative induction principle for $\mathbf{Z}_h$ only specifies the behavior of the dependent eliminator on 0_h and succ_h.
- More generally, $\mathbb{Z}$-algebra morphisms only carry witnesses of commutativity with the underlying maps of the relevant autoequivalences.
- It is **not** true in general that we may ignore preservation of properties (see Capriotti's counterexample in [1]).
- However, we can prove that, in this particular case, we may. The missing information is fully determined by the existing data and can thus be recovered if necessary.

Theorem

For every $(s_A, p_A, \sigma_A, \rho_A, \tau_A) : A_1 \simeq A_2$, $(s_B, p_B, \sigma_B, \rho_B, \tau_B) : B_1 \simeq B_2$, $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ *and* $f_s : f_2 \circ s_A \sim s_B \circ f_1$, *the type of quadruples with components*

- $f_p : f_1 \circ p_A \sim p_B \circ f_2$,
- $f_\sigma : f_1 \circ \sigma_A \sim \mathsf{top}_\sigma \cdot (\sigma_B \circ f_1)$,
- $f_\rho : f_2 \circ \rho_A \sim \mathsf{top}_\rho \cdot (\rho_B \circ f_2)$,
- $f_\tau : (f_2 \circ \tau_A) \cdot_r f_s) \cdot \mathsf{back} \sim \mathsf{front} \cdot (\mathsf{top}_\tau \cdot_l (\tau_B \circ f_1)$,

is contractible.

Informally: the corresponding "functor" into the category of $\mathbb{Z}$-algebras is fully faithful.

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Proof.

Apply equivalence and homotopy induction. The resulting $\sum$-type contains several summands that are easily seen to be contractible (essentially, paths with a free endpoint). $\square$

Additional Results

Some of the ideas that we have presented apply to any naïve category $\mathcal{C}$ with finite limits.

Definition

An object A in $\mathcal{C}$ is **inductive** if every morphism into it has a section, i.e. if the type

$$\mathrm{isind}(A) :\equiv \prod_{B:\mathcal{C}_0} \prod_{f:\mathrm{Hom}_{\mathcal{C}}(B,A)} \sum_{g:\mathrm{Hom}_{\mathcal{C}}(A,B)} f \circ g = 1_A$$

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is inhabited.

The following is a standard fact in classical category theory:

Theorem

For every $A : \mathcal{C}_0$, $\mathrm{isind}(A) \leftrightarrow \mathrm{ishinit}(A)$.

Additional Results

We can generalize it slightly as follows:

Theorem (Uniqueness Principle for Inductive Objects)

If $A : \mathcal{C}_0$ is inductive, then, for any $B : \mathcal{C}_0$ and $f : \mathrm{Hom}_{\mathcal{C}}(B, A)$, the type

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Corollary

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References I

[1] Thorsten Altenkirch and Luis Scoccola. "The Integers as a Higher Inductive Type". In: *arXiv e-prints*, arXiv:2007.00167 (June 2020), arXiv:2007.00167. arXiv: 2007.00167 [cs.LO].

[2] Steve Awodey, Nicola Gambino, and Kristina Sojakova. "Homotopy-Initial Algebras in Type Theory". In: *J. ACM* 63.6 (Jan. 2017). ISSN: 0004-5411. DOI: 10.1145/3006383. URL: https://doi.org/10.1145/3006383.

[3] Evan Cavallo. *biinv-int*. URL: https://github.com/RedPRL/redtt/blob/master/library/cool/biinv-int.red.

[4] Egbert Rijke. *Introduction to Homotopy Type Theory*. URL: https://github.com/EgbertRijke/HoTT-Intro.

References II

[5] Kristina Sojakova. "Higher Inductive Types as Homotopy-Initial Algebras". In: *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. POPL '15. Mumbai, India: Association for Computing Machinery, 2015, pp. 31–42. ISBN: 9781450333009. DOI: 10.1145/2676726.2676983. URL: https://doi.org/10.1145/2676726.2676983.

[6] Kristina Sojakova. "Higher Inductive Types as Homotopy-Initial Algebras". PhD thesis. Carnegie Mellon University, 2016.