Sphere Bundles and Characteristic Classes

Wes Caldwell

Carnegie Mellon University

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Fiber Bundles (classical)

A *fiber bundle* is a continuous surjection $p\colon E \to B$ with a local trivialization. That is, for each point $x \in B$, there is an open neighborhood $U \subseteq B$ of x such that $p^{-1}(U)$ is homeomorphic to $U \times p^{-1}(\{x\})$.

If B is connected, then certainly there is some space F such that $p^{-1}(x) \cong F$ for all $x \in B$; F is called the *fiber*.

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- $E \cong F \times B \Rightarrow p\colon E \to B$ is a *trivial bundle*.
- $F \cong \mathbb{S}^k \Rightarrow p\colon E \to B$ is a *sphere bundle*.
- F is discrete space $\Rightarrow p\colon E \to B$ is a *covering projection*.
- $F \cong \mathbb{R}^k \Rightarrow p\colon E \to B$ is a (real) *vector bundle*.
- $p\colon E \to B$ is compatible with a free & transitive action of a group G on $E \Rightarrow p$ is a *principal G-bundle*.

Constructions on Bundles

A *bundle map* from $p\colon E \to X$ to $q\colon F \to Y$ is a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ X & \longrightarrow & Y \end{array}$$

Given a map $f\colon X \to Y$ and a fiber bundle $q\colon F \to Y$, we can construct the *pullback bundle* $f^*q\colon f^*F \to X$ simply by taking the pullback

$$\begin{array}{ccc} f^*F & \longrightarrow & F \\ \big\downarrow{\scriptstyle f^*q} & & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

Therefore, b: Top $\to$ Set taking a space X to the set of isomorphism classes of bundles over X is a contravariant functor.

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Characteristic Classes

A *characteristic class* c of bundles is a natural transformation from b to a cohomology functor H^* (forgetting group structure). In other words, a characteristic class c associates to each bundle $E \to X$ an element $c(E) \in H^*(X)$ such that, if $f\colon Y \to X$ is a continuous map, then $c(f^*E) = f^*c(E)$.

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Characteristic classes are most commonly defined for vector bundles, but some only depend on the corresponding unit sphere bundle of the vector bundle, so this will not be a problem when formulating these concepts in homotopy type theory.

Sections

A *section* of the fiber bundle $p\colon E \to B$ is a map $s\colon B \to E$ that is right inverse to p. For example,

- Every vector bundle $p\colon E \to B$ admits at least one section, the *zero section*.
- A section of the tangent bundle $p\colon TM \to M$ is a vector field.
- A section of the cotangent bundle $p\colon T^*M \to M$ is a 1-form.

The existence of sections satisfying certain properties are often of great importance; for instance, the hairy ball theorem shows that there are no nonvanishing vector fields on even-dimensional spheres.

Homotopy Type Theory

The direct analogue of the classical notion of fiber bundle in homotopy type theory would appear to be a surjection $p\colon E \to B$; however, since we have the equivalence

$$\begin{aligned} E &\simeq \sum_{y:E} \mathbf{1} \\ &\simeq \sum_{y:E} \sum_{x:B} p(y) = x \\ &\simeq \sum_{x:B} \mathrm{fib}_p(x), \end{aligned}$$

it is more natural to consider the definition of a fiber bundle to be a type family $P\colon B \to \mathcal{U}$ and define the total space to be $E \simeq \sum_{x:B} P(x)$. If B is 0-connected, then the fiber $P(x)$ is independent of choice of x.

Given $f : C \to B$, we can construct the pullback bundle f^*E:

$$\begin{aligned} f^*E &\equiv \sum_{x:C} \sum_{y:E} f(x) = p(y) \\ &\simeq \sum_{x:C} \sum_{y:B} \sum_{z:P(y)} f(x) = y \\ &\simeq \sum_{x:C} \sum_{y:B} P(y) \times (f(x) = y) \\ &\simeq \sum_{x:C} \sum_{y:B} P(f(x)) \times (f(x) = y) \\ &\simeq \sum_{x:C} P(f(x)) \times \sum_{y:B} (f(x) = y) \\ &\simeq \sum_{x:C} P(f(x)), \end{aligned}$$

so the pullback bundle f^*P is just the precomposition $P \circ f$!

Sections

Conceptually, sections give elements of the fiber that "lie above" the base space B. In HoTT, we can view this as a dependent function type:

$$s\colon \prod_{b:B} P(b).$$

Then we can recover the classical notion of section as a right inverse to projection by defining the map $\overline{s}\colon B \to E, b \mapsto (b, s(b))$.

Let B be a pointed, 0-connected type. Then

- A *fiber bundle* over B is of type $B \to \mathcal{U}$.
- A *covering space* over B is of type $B \to \mathsf{Set}$.
- An *n-sphere bundle* over B is of type $\sum_{P:B\to\mathcal{U}} P(*_B) \simeq \mathbb{S}^n$.
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- A *principal* G-*bundle* over B is ???

We will focus on the case of sphere bundles, denoting

$$n\text{-}\mathsf{Sph}(X) :\equiv \sum_{P:B\to\mathcal{U}} P(*_B) \simeq \mathbb{S}^n$$

the type of n-sphere bundles.

Cohomology

The final ingredient remaining to define characteristic classes is cohomology. With the machinery of Eilenberg-MacLane spaces (Licata, Finster '14), this is no problem.

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The (unreduced) nth cohomology group of X with coefficients in G is given by

$$H^n(X;G) :\equiv \|X \to K(G,n)\|_0,$$

with reduced cohomology groups given by

$$\widetilde{H}^n(X;G) :\equiv \|X \to_* K(G,n)\|_0.$$

The group operations are simply lifted from the n-cell concatenation in $K(G,n)$, which is itself lifted from the operation of G.

The Thom Class

Let $P\colon B \to \mathcal{U}$ be any bundle. Then the *Thom space* $\mathrm{Th}(P)$ is the (homotopy) cofiber of the natural projection $\pi\colon E \to B$:

$$\begin{array}{ccc} \sum_{b:B} P(b) & \longrightarrow & \mathbf{1} \\ \downarrow{\scriptstyle \pi} & \nearrow\!\!\!\nearrow & \downarrow \\ B & \longrightarrow & \mathrm{Th}(P) \end{array}$$

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Equivalently, $\mathrm{Th}(P)$ is the higher inductive type with the following constructors:

$$\begin{aligned} *_{\mathrm{Th}}&\colon \mathrm{Th}(P) \\ i&\colon B \to \mathrm{Th}(P) \\ \mathrm{glue}&\colon \prod_{b\colon B} P(b) \to (i(b) = *_{\mathrm{Th}}) \end{aligned}$$

For each $b : B$, we have the ingredients to define the cocone

$$\begin{array}{ccc} P(b) & \longrightarrow & \mathbf{1} \\ \big\downarrow & \overset{\mathrm{glue}(b)}{\nearrow\!\!\!\!\nearrow} & \big\downarrow {\scriptstyle *_{\mathrm{Th}}} \\ \mathbf{1} & \xrightarrow{\;i(b)\;} & \mathrm{Th}(P) \end{array}$$

which furnishes a map $s_b \colon \Sigma P(b) \to \mathrm{Th}(P)$ for each $b : B$ by the universal property of suspension.

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Thom classes

Fix an $(n-1)$-sphere bundle $P \colon B \to \mathcal{U}$. Then a *Thom class* is a cohomology class $c \in H^n(\mathrm{Th}(P))$ such that for all $b : B$, $s_b^* c \in H^n(\Sigma P(b)) \simeq H^n(\mathbb{S}^n) \simeq \mathbb{Z}$ is the same generator (± 1).

If the Thom class exists, then the maps s_b glue together into $s \colon \prod_{b:B} \Sigma P(b) \to \mathrm{Th}(P)$.

Not all sphere bundles have a Thom class! For instance, the Klein bottle, considered as a 1-sphere bundle over S^1, does not have a Thom class; classically, we would say that the Klein bottle is *nonorientable*. In fact, we can take the existence of a Thom class as the **definition** of orientability.

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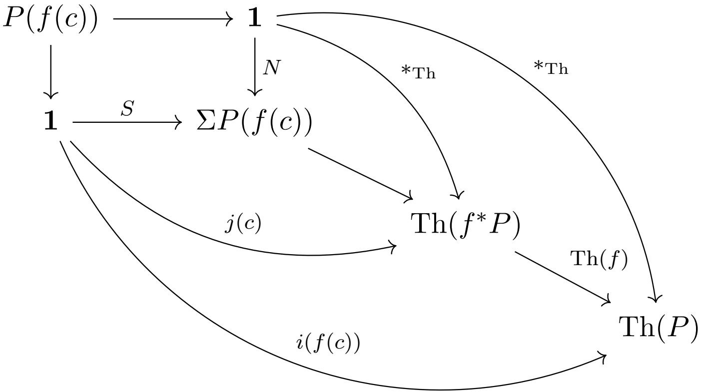
The Thom class satisfies the following functoriality theorem:

Functoriality of the Thom class

Given an oriented $(n-1)$-sphere bundle $P\colon B \to \mathcal{U}$ and a function $f\colon C \to B$, there is an induced map $\mathrm{Th}(f)\colon \mathrm{Th}(f^*P) \to \mathrm{Th}(P)$ which pulls back the Thom class of P to the Thom class of f^*P.

This theorem has an elegant proof in the homotopy type theoretic formulation.

For each $c : C$, consider the commutative diagram



Since $\Sigma P(f(c)) \to \mathrm{Th}(P)$ necessarily factors through $\mathrm{Th}(f^*P)$ by universality, $\mathrm{Th}(f)$ pulls back to the Thom class.

The defining pushout of the Thom space factors through the pushout

$$
\begin{array}{ccccc}
\sum_{b:B} P(b) & \xrightarrow{\pi_1} & B & \longrightarrow & \mathbf{1} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle b \mapsto (b,N)} & & \downarrow{\scriptstyle *_{\mathrm{Th}}} \\
B & \xrightarrow{b \mapsto (b,S)} & \sum_{b:B} \Sigma P(b) & \longrightarrow & \mathrm{Th}(P)
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With some work, this induces the *Thom diagonal* $\Delta\colon \mathrm{Th}(P) \to B_+ \wedge \mathrm{Th}(P)$, which lets us define a cup product

$$\smile\colon H^p(B) \otimes \tilde{H}^q(\mathrm{Th}(P)) \to \tilde{H}^{p+q}(\mathrm{Th}(P)).$$

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Thom Isomorphism

Let $c \in \tilde{H}^n(\mathrm{Th}(P))$ be a Thom class of the $(n-1)$-sphere bundle $P \colon B \to \mathcal{U}$. Then the map $\Phi \colon H^k(B) \to \tilde{H}^{k+n}(\mathrm{Th}(P))$ defined by $\Phi(x) :\equiv x \smile c$ is an isomorphism of groups.

The Euler class

Let $P\colon B \to \mathcal{U}$ be an oriented $(n-1)$-sphere bundle. Then the *Euler class* is the pullback of the Thom class $c \in H^n(\mathrm{Th}(P))$ by the inclusion map $i\colon B \to \mathrm{Th}(P)$:

$$e(P) :\equiv i^*c \in H^n(B).$$

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The Euler class satisfies 3 key properties:

- **Functoriality:** Given $f\colon C \to B$, the Euler class of the pullback bundle satisfies $e(f^*P) = f^*e(P)$.
- **Orientation:** If $\overline{P}\colon B \to \mathcal{U}$ is P with the opposite orientation, then $e(\overline{P}) = -e(P)$.
- **Normalization:** If P admits a section $s\colon \prod_{b:B} P(b)$, then $e(P) = 0$.

The Gysin Sequence

Since the Thom space is the cofiber of the total space projection $\pi\colon E \to B$, it has an associated exact sequence

$$\cdots \to H^{k-1}(E) \to H^k(\mathrm{Th}(P)) \xrightarrow{i^*} H^k(B) \xrightarrow{\pi^*} H^k(E) \to \dots$$

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Applying the Thom isomorphism and simplifying the corresponding maps, we get the *Gysin sequence*

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We don't actually need for $P(b)$ to be a sphere; we merely need that $P(b)$ is a (co)homology sphere!

Rational Homotopy Groups of Spheres

Serre Finiteness Theorem

The homotopy groups of spheres are finite with the exception of

$$\pi_n(S^n) \simeq \mathbb{Z}$$
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Proof sketch:

- Localize spheres with respect to degree maps to construct $\mathbb{S}^n_{\mathbb{Q}}$, i.e. types with $\pi_k(\mathbb{S}^n_{\mathbb{Q}}) \simeq \pi_k(\mathbb{S}^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Calculate the cohomology groups of $K(\mathbb{Q}, n)$.
- Study the connectedness of the truncation maps $\tau_n \colon \mathbb{S}^n_{\mathbb{Q}} \to K(\mathbb{Q}, n)$ to move from cohomology back to homotopy.

Rational Types (Christensen, et al '18)

A type $X : \mathcal{U}$ is *rational* if it is deg-local; that is,

$$(- \circ \mathsf{deg}_n) : (\mathbb{S}^1 \to X) \to (\mathbb{S}^1 \to X)$$

are equivalences for all $n : \mathbb{N}$.

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Rationalization

A *rationalization* of X is a type $X' : \mathcal{U}$ and a map $g \colon X \to X'$ such that for every rational type Y, the precomposition map

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We denote the rationalization of the n-sphere by $\mathbb{S}^n_{\mathbb{Q}}$, and note that $\pi_k(\mathbb{S}^n_{\mathbb{Q}}) \simeq \pi_k(\mathbb{S}^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Cohomology of Rational Eilenberg-MacLane Spaces

If n is even, then $H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[e]$, the polynomial algebra over $\mathbb{Q}$ with generator $e : H^n(K(\mathbb{Q}, n))$.
If n is odd, then $H^*(K(\mathbb{Q}, n))$ is concentrated in dimension n, i.e. $K(\mathbb{Q}, n)$ is an n-(co)homology sphere.

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- If n is even, then by hypothesis $K(\mathbb{Q}, n-1)$ is a cohomology sphere and we can apply the Gysin sequence to the path fibration $K(\mathbb{Q}, n-1) \simeq \Omega K(\mathbb{Q}, n) \to \mathbf{1} \to K(\mathbb{Q}, n)$.

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- If n is odd, we analyze the fiber sequence of the truncation $\tau_n \colon \mathbb{S}^n_{\mathbb{Q}} \to K(\mathbb{Q}, n)$.

$$\cdots \to K(\mathbb{Q}, n-1) \to \mathrm{fib}_{\tau_n}(*_{K(\mathbb{Q},n)}) \to \mathbb{S}^n_{\mathbb{Q}} \to K(\mathbb{Q}, n)$$

Sketch of the Finiteness Theorem

For n odd:

- The n-truncation $\tau_n \colon \mathbb{S}^n_{\mathbb{Q}} \to K(\mathbb{Q}, n)$ induces an equivalence in cohomology, so its n-connected fiber has trivial cohomology.

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- Use the Postnikov system for the fiber to show that $\mathrm{Hom}(\pi_{n+1}(F), \mathbb{Q}) = 0$.

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- Appeal to the finite generation of homotopy groups† to conclude that the fiber is in fact $(n+1)$-connected.

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- Appeal to the finite generation of homotopy groups[†] to conclude that the fiber is in fact $(n+1)$-connected.
- By induction, therefore the fiber has trivial homotopy groups.
- Conclude that by the long exact sequence of τ_n, we get that $\pi_*(\mathbb{S}^n_{\mathbb{Q}}) \simeq \pi_*(K(\mathbb{Q}, n))$ is localized in degree n.

For n even:

- Use the Gysin sequence associated to the fiber sequence $K(\mathbb{Q}, n-1) \to \mathrm{fib}_{\tau_n}(*_{K(\mathbb{Q},n)}) \xrightarrow{j} \mathbb{S}^n_{\mathbb{Q}}$ to show that $\mathrm{fib}_{\tau_n}(*)$ is a $(2n-1)$-cohomology sphere.

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- By the universal property of rationalization, since $\mathrm{fib}_{\tau_n}(*)$ is a rational type, there is an induced map $f_{\mathbb{Q}}\colon \mathbb{S}^{2n-1}_{\mathbb{Q}} \to \mathrm{fib}_{\tau_n}(*)$.

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- By the universal property of rationalization, since $\mathrm{fib}_{\tau_n}(*)$ is a rational type, there is an induced map $f_{\mathbb{Q}}\colon \mathbb{S}^{2n-1}_{\mathbb{Q}} \to \mathrm{fib}_{\tau_n}(*)$.
- $f_{\mathbb{Q}}$ induces an isomorphism in cohomology, so therefore also induces an isomorphism in homotopy as well (examine the fiber).

For n even:

- Use the Gysin sequence associated to the fiber sequence $K(\mathbb{Q}, n-1) \to \mathrm{fib}_{\tau_n}(*_{K(\mathbb{Q},n)}) \xrightarrow{j} \mathbb{S}^n_{\mathbb{Q}}$ to show that $\mathrm{fib}_{\tau_n}(*)$ is a $(2n-1)$-cohomology sphere.
- Since $\pi_{2n-1}(\mathrm{fib}_{\tau_n}(*)) \simeq \pi_{2n-1}(\mathbb{S}^n_{\mathbb{Q}})$ is nontrivial (Brunerie, 2016), there is a nontrivial map $f\colon \mathbb{S}^{2n-1} \to \mathrm{fib}_{\tau_n}(*)$.
- By the universal property of rationalization, since $\mathrm{fib}_{\tau_n}(*)$ is a rational type, there is an induced map $f_{\mathbb{Q}}\colon \mathbb{S}^{2n-1}_{\mathbb{Q}} \to \mathrm{fib}_{\tau_n}(*)$.
- $f_{\mathbb{Q}}$ induces an isomorphism in cohomology, so therefore also induces an isomorphism in homotopy as well (examine the fiber).
- Therefore, for $i \neq n$,

$$\pi_i(\mathbb{S}^n_{\mathbb{Q}}) \simeq \pi_i(\mathrm{fib}_{\tau_n}) \simeq \pi_i(\mathbb{S}^{2n-1}_{\mathbb{Q}}),$$

  so the homotopy groups $\pi_i(\mathbb{S}^n_{\mathbb{Q}})$ are localized at $i = n$ and $i = 2n-1$.

Reference

Classical References:

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