Sphere Bundles and Characteristic Classes

Wes Caldwell

Carnegie Mellon University

26 February 2021
Fiber Bundles (classical)

A fiber bundle is a continuous surjection \( p : E \rightarrow B \) with a local trivialization. That is, for each point \( x \in B \), there is an open neighborhood \( U \subseteq B \) of \( x \) such that \( p^{-1}(U) \) is homeomorphic to \( U \times p^{-1}(\{x\}) \).

If \( B \) is connected, then certainly there is some space \( F \) such that \( p^{-1}(x) \cong F \) for all \( x \in B \); \( F \) is called the fiber.
Fiber Bundles (classical)

A fiber bundle is a continuous surjection $p: E \to B$ with a local trivialization. That is, for each point $x \in B$, there is an open neighborhood $U \subseteq B$ of $x$ such that $p^{-1}(U)$ is homeomorphic to $U \times p^{-1}({x})$.

If $B$ is connected, then certainly there is some space $F$ such that $p^{-1}(x) \cong F$ for all $x \in B$; $F$ is called the fiber.

- $E \cong F \times B \Rightarrow p: E \to B$ is a trivial bundle.
- $F \cong \mathbb{S}^k \Rightarrow p: E \to B$ is a sphere bundle.
- $F$ is discrete space $\Rightarrow p: E \to B$ is a covering projection.
- $F \cong \mathbb{R}^k \Rightarrow p: E \to B$ is a (real) vector bundle.
- $p: E \to B$ is compatible with a free & transitive action of a group $G$ on $E \Rightarrow p$ is a principal $G$-bundle.
Constructions on Bundles

A *bundle map* from $p: E \to X$ to $q: F \to Y$ is a commutative diagram

\[
\begin{array}{ccc}
E & \longrightarrow & F \\
\downarrow p & & \downarrow q \\
X & \longrightarrow & Y
\end{array}
\]

Given a map $f: X \to Y$ and a fiber bundle $q: F \to Y$, we can construct the *pullback bundle* $f^*q: f^*F \to X$ simply by taking the pullback

\[
\begin{array}{ccc}
f^*F & \longrightarrow & F \\
\downarrow f^*q & & \downarrow q \\
X & \underset{f}{\longrightarrow} & Y
\end{array}
\]

Sphere Bundles and Characteristic Classes
Wes Caldwell
Therefore, \( b: \text{Top} \to \text{Set} \) taking a space \( X \) to the set of isomorphism classes of bundles over \( X \) is a contravariant functor.
Therefore, \( b : \text{Top} \to \text{Set} \) taking a space \( X \) to the set of isomorphism classes of bundles over \( X \) is a contravariant functor.

### Characteristic Classes

A *characteristic class* \( c \) of bundles is a natural transformation from \( b \) to a cohomology functor \( H^* \) (forgetting group structure). In other words, a characteristic class \( c \) associates to each bundle \( E \to X \) an element \( c(E) \in H^*(X) \) such that, if \( f : Y \to X \) is a continuous map, then \( c(f^*E) = f^*c(E) \).
Therefore, $b : \text{Top} \to \text{Set}$ taking a space $X$ to the set of isomorphism classes of bundles over $X$ is a contravariant functor.

### Characteristic Classes

A characteristic class $c$ of bundles is a natural transformation from $b$ to a cohomology functor $H^*$ (forgetting group structure). In other words, a characteristic class $c$ associates to each bundle $E \to X$ an element $c(E) \in H^*(X)$ such that, if $f : Y \to X$ is a continuous map, then $c(f^*E) = f^*c(E)$.

Characteristic classes are most commonly defined for vector bundles, but some only depend on the corresponding unit sphere bundle of the vector bundle, so this will not be a problem when formulating these concepts in homotopy type theory.
Sections

A *section* of the fiber bundle $p: E \to B$ is a map $s: B \to E$ that is right inverse to $p$. For example,

- Every vector bundle $p: E \to B$ admits at least one section, the *zero section*.
- A section of the tangent bundle $p: TM \to M$ is a vector field.
- A section of the cotangent bundle $p: T^*M \to M$ is a 1-form.

The existence of sections satisfying certain properties are often of great importance; for instance, the hairy ball theorem shows that there are no nonvanishing vector fields on even-dimensional spheres.
Homotopy Type Theory

The direct analogue of the classical notion of fiber bundle in homotopy type theory would appear to be a surjection $p: E \rightarrow B$; however, since we have the equivalence

$$E \simeq \sum_{y:E} 1 \simeq \sum_{y:E} \sum_{x:B} p(y) = x \simeq \sum_{x:B} \text{fib}_p(x),$$

it is more natural to consider the definition of a fiber bundle to be a type family $P: B \rightarrow \mathcal{U}$ and define the total space to be $E \simeq \sum_{x:B} P(x)$. If $B$ is 0-connected, then the fiber $P(x)$ is independent of choice of $x$. 
Given $f : C \to B$, we can construct the pullback bundle $f^* E$:

\[
f^* E \equiv \sum_{x:C} \sum_{y:E} f(x) = p(y)
\]

\[
\simeq \sum_{x:C} \sum_{y:B} \sum_{z:P(y)} f(x) = y
\]

\[
\simeq \sum_{x:C} \sum_{y:B} P(y) \times (f(x) = y)
\]

\[
\simeq \sum_{x:C} \sum_{y:B} P(f(x)) \times (f(x) = y)
\]

\[
\simeq \sum_{x:C} P(f(x)) \times \sum_{y:B} (f(x) = y)
\]

\[
\simeq \sum_{x:C} P(f(x)),
\]

so the pullback bundle $f^* P$ is just the precomposition $P \circ f$!
Sections

Conceptually, sections give elements of the fiber that "lie above" the base space $B$. In HoTT, we can view this as a dependent function type:

$$s : \prod_{b:B} P(b).$$

Then we can recover the classical notion of section as a right inverse to projection by defining the map $\bar{s} : B \to E, b \mapsto (b, s(b))$. 
Let $B$ be a pointed, 0-connected type. Then

- A fiber bundle over $B$ is of type $B \to \mathcal{U}$.
- A covering space over $B$ is of type $B \to \text{Set}$.
- An $n$-sphere bundle over $B$ is of type $\sum_{P : B \to \mathcal{U}} P(*_B) \simeq \mathbb{S}^n$.
- A principal $G$-bundle over $B$ is ???

We will focus on the case of sphere bundles, denoting $n$-Sph$(X) : \equiv \sum_{P : B \to \mathcal{U}} P(*_B) \simeq \mathbb{S}^n$ the type of $n$-sphere bundles.
Let $B$ be a pointed, 0-connected type. Then

- A fiber bundle over $B$ is of type $B \to \mathcal{U}$.
- A covering space over $B$ is of type $B \to \text{Set}$.
- An $n$-sphere bundle over $B$ is of type $\sum_{P : B \to \mathcal{U}} P(*_B) \simeq \mathbb{S}^n$.
- A principal $G$-bundle over $B$ is ???

We will focus on the case of sphere bundles, denoting

$$n\text{-Sph}(X) \equiv \sum_{P : B \to \mathcal{U}} P(*_B) \simeq \mathbb{S}^n$$

the type of $n$-sphere bundles.
Cohomology

The final ingredient remaining to define characteristic classes is cohomology. With the machinery of Eilenberg-MacLane spaces (Licata, Finster ’14), this is no problem.
Cohomology

The final ingredient remaining to define characteristic classes is cohomology. With the machinery of Eilenberg-MacLane spaces (Licata, Finster ’14), this is no problem. The (unreduced) $n$th cohomology group of $X$ with coefficients in $G$ is given by

$$H^n(X; G) :\equiv \| X \rightarrow K(G, n) \|_0,$$

with reduced cohomology groups given by

$$\tilde{H}^n(X; G) :\equiv \| X \rightarrow_* K(G, n) \|_0.$$

The group operations are simply lifted from the $n$-cell concatenation in $K(G, n)$, which is itself lifted from the operation of $G$. 
The Thom Class

Let $P: B \to \mathcal{U}$ be any bundle. Then the Thom space $\text{Th}(P)$ is the (homotopy) cofiber of the natural projection $\pi: E \to B$:

$$
\sum_{b:B} P(b) \xrightarrow{\pi} 1 \\
\downarrow \quad \uparrow \\
B \quad \quad \text{Th}(P)
$$
The Thom Class

Let $P : B \to \mathcal{U}$ be any bundle. Then the Thom space $\text{Th}(P)$ is the (homotopy) cofiber of the natural projection $\pi : E \to B$:

$$\sum_{b : B} P(b) \longrightarrow 1$$

$$\begin{array}{c}
\downarrow \pi \\
B \longrightarrow \text{Th}(P)
\end{array}$$

Equivalently, $\text{Th}(P)$ is the higher inductive type with the following constructors:

$$\begin{align*}
*_{\text{Th}} &: \text{Th}(P) \\
i &: B \to \text{Th}(P) \\
glue &: \prod_{b : B} P(b) \to (i(b) = *_{\text{Th}})
\end{align*}$$
For each \( b : B \), we have the ingredients to define the cocone

\[
\begin{array}{ccc}
P(b) & \xrightarrow{\text{glue}(b)} & 1 \\
\downarrow & \nearrow & \downarrow^{*_{\text{Th}}} \\
1 & \xrightarrow{i(b)} & \text{Th}(P)
\end{array}
\]

which furnishes a map \( s_b : \Sigma P(b) \to \text{Th}(P) \) for each \( b : B \) by the universal property of suspension.
For each \( b : B \), we have the ingredients to define the cocone

\[
\begin{array}{ccc}
P(b) & \longrightarrow & 1 \\
\downarrow & & \downarrow^{\ast \text{Th}} \\
1 & \longrightarrow & \text{Th}(P)
\end{array}
\]

which furnishes a map \( s_b : \Sigma P(b) \to \text{Th}(P) \) for each \( b : B \) by the universal property of suspension.

**Thom classes**

Fix an \((n-1)\)-sphere bundle \( P : B \to \mathcal{U} \). Then a *Thom class* is a cohomology class \( c \in H^n(\text{Th}(P)) \) such that for all \( b : B \),

\[
s_b^* c \in H^n(\Sigma P(b)) \cong H^n(S^n) \cong \mathbb{Z}
\]

is the same generator \((\pm 1)\).

If the Thom class exists, then the maps \( s_b \) glue together into \( s : \prod_{b : B} \Sigma P(b) \to \text{Th}(P) \).
Not all sphere bundles have a Thom class! For instance, the Klein bottle, considered as a 1-sphere bundle over $S^1$, does not have a Thom class; classically, we would say that the Klein bottle is \emph{nonorientable}. In fact, we can take the existence of a Thom class as the \textbf{definition} of orientability.
Not all sphere bundles have a Thom class! For instance, the Klein bottle, considered as a 1-sphere bundle over $S^1$, does not have a Thom class; classically, we would say that the Klein bottle is nonorientable. In fact, we can take the existence of a Thom class as the definition of orientability.

The Thom class satisfies the following functoriality theorem:

**Functoriality of the Thom class**

Given an oriented $(n - 1)$-sphere bundle $P: B \to \mathcal{U}$ and a function $f: C \to B$, there is an induced map $\text{Th}(f): \text{Th}(f^*P) \to \text{Th}(P)$ which pulls back the Thom class of $P$ to the Thom class of $f^*P$.

This theorem has an elegant proof in the homotopy type theoretic formulation.
For each $c : C$, consider the commutative diagram

$$P(f(c)) \rightarrow 1 \rightarrow \Sigma P(f(c))$$

Since $\Sigma P(f(c)) \rightarrow \text{Th}(P)$ necessarily factors through $\text{Th}(f^*P)$ by universality, $\text{Th}(f)$ pulls back to the Thom class.
The defining pushout of the Thom space factors through the pushout

$$
\begin{array}{ccc}
\sum_{b:B} P(b) & \xrightarrow{\pi_1} & B \\
\downarrow{\pi_1} & & \downarrow{b \mapsto (b,N)} & \mapsto & \downarrow{\ast_{\text{Th}}} \\
B & \xrightarrow{b \mapsto (b,S)} & \sum_{b:B} \Sigma P(b) & \rightarrow & \text{Th}(P)
\end{array}
$$
The defining pushout of the Thom space factors through the pushout

\[
\begin{array}{ccc}
\sum_{b:B} P(b) & \xrightarrow{\pi_1} & B & \xrightarrow{\pi_1} & \sum_{b:B} \Sigma P(b) & \xrightarrow{*\text{Th}} & \text{Th}(P)
\end{array}
\]

With some work, this induces the *Thom diagonal* 
\[ \Delta: \text{Th}(P) \to B_+ \wedge \text{Th}(P), \] which lets us define a cup product

\[ \cup: \tilde{H}^p(B) \otimes \tilde{H}^q(\text{Th}(P)) \to \tilde{H}^{p+q}(\text{Th}(P)). \]
The defining pushout of the Thom space factors through the pushout

\[
\sum_{b:B} P(b) \xrightarrow{\pi_1} B \xrightarrow{\pi_1} 1
\]

\[
B \xrightarrow{b \mapsto (b,S)} \sum_{b:B} \sum P(b) \xrightarrow{} \text{Th}(P)
\]

With some work, this induces the *Thom diagonal* \(\Delta: \text{Th}(P) \to B_+ \wedge \text{Th}(P)\), which lets us define a cup product

\[
\smile: H^p(B) \otimes \tilde{H}^q(\text{Th}(P)) \to \tilde{H}^{p+q}(\text{Th}(P)).
\]

**Thom Isomorphism**

Let \(c \in \tilde{H}^n(\text{Th}(P))\) be a Thom class of the \((n - 1)\)-sphere bundle \(P: B \to U\). Then the map \(\Phi: H^k(B) \to \tilde{H}^{k+n}(\text{Th}(P))\) defined by \(\Phi(x) \equiv x \smile c\) is an isomorphism of groups.
The Euler class

Let $P : B \to \mathcal{U}$ be an oriented $(n - 1)$-sphere bundle. Then the **Euler class** is the pullback of the Thom class $c \in H^n(\text{Th}(P))$ by the inclusion map $i : B \to \text{Th}(P)$:

$$e(P) \equiv i^*c \in H^n(B).$$
The Euler class

Let \( P: B \to \mathcal{U} \) be an oriented \((n - 1)\)-sphere bundle. Then the \textit{Euler class} is the pullback of the Thom class \( c \in H^n(\text{Th}(P)) \) by the inclusion map \( i: B \to \text{Th}(P) \):

\[
e(P) \equiv i^*c \in H^n(B).
\]

The Euler class satisfies 3 key properties:

- **Functoriality:** Given \( f: C \to B \), the Euler class of the pullback bundle satisfies \( e(f^*P) = f^*e(P) \).

- **Orientation:** If \( \overline{P}: B \to \mathcal{U} \) is \( P \) with the opposite orientation, then \( e(\overline{P}) = -e(P) \).

- **Normalization:** If \( P \) admits a section \( s: \prod_{b:B} P(b) \), then \( e(P) = 0 \).
The Gysin Sequence

Since the Thom space is the cofiber of the total space projection $\pi: E \to B$, it has an associated exact sequence

$$
\cdots \to H^{k-1}(E) \to H^k(\text{Th}(P)) \xrightarrow{i^*} H^k(B) \xrightarrow{\pi^*} H^k(E) \to \cdots
$$

Applying the Thom isomorphism and simplifying the corresponding maps, we get the Gysin sequence

$$
\cdots \to H^{k-1}(E) \to H^k(B) \xrightarrow{\pi^*} H^k(E) \to \cdots
$$

where $e: H^n(B)$ is the Euler class.

We don't actually need for $P(b)$ to be a sphere; we merely need that $P(b)$ is a (co)homology sphere!
The Gysin Sequence

Since the Thom space is the cofiber of the total space projection \( \pi: E \to B \), it has an associated exact sequence

\[
\cdots \to H^{k-1}(E) \to H^k(\text{Th}(P)) \xrightarrow{i^*} H^k(B) \xrightarrow{\pi^*} H^k(E) \to \cdots
\]

Applying the Thom isomorphism and simplifying the corresponding maps, we get the \textit{Gysin sequence}

\[
\cdots \to H^{k-1}(E) \to H^{k-n}(B) \xrightarrow{\sim e} H^k(B) \xrightarrow{\pi^*} H^k(E) \to \cdots
\]

where \( e: H^n(B) \) is the Euler class.
The Gysin Sequence

Since the Thom space is the cofiber of the total space projection \( \pi : E \to B \), it has an associated exact sequence

\[
\cdots \to H^{k-1}(E) \to H^k(\text{Th}(P)) \overset{i^*}{\to} H^k(B) \overset{\pi^*}{\to} H^k(E) \to \cdots
\]

Applying the Thom isomorphism and simplifying the corresponding maps, we get the Gysin sequence

\[
\cdots \to H^{k-1}(E) \to H^{k-n}(B) \overset{\sim e}{\to} H^k(B) \overset{\pi^*}{\to} H^k(E) \to \cdots
\]

where \( e : H^n(B) \) is the Euler class.

We don’t actually need for \( P(b) \) to be a sphere; we merely need that \( P(b) \) is a (co)homology sphere!
Rational Homotopy Groups of Spheres

**Serre Finiteness Theorem**

The homotopy groups of spheres are finite with the exception of

\[ \pi_n(S^n) \cong \mathbb{Z} \]

\[ \pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \text{torsion} \]
Rational Homotopy Groups of Spheres

Serre Finiteness Theorem

The homotopy groups of spheres are finite with the exception of

\[ \pi_n(S^n) \cong \mathbb{Z} \]

\[ \pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \text{torsion} \]

Proof sketch:

- Localize spheres with respect to degree maps to construct \( S^n_Q \), i.e. types with \( \pi_k(S^n_Q) \cong \pi_k(S^n) \otimes \mathbb{Z} \mathbb{Q} \).
- Calculate the cohomology groups of \( K(\mathbb{Q}, n) \).
- Study the connectedness of the truncation maps \( \tau_n : S^n_Q \rightarrow K(\mathbb{Q}, n) \) to move from cohomology back to homotopy.
Rational Types (Christensen, et al '18)

A type $X : \mathcal{U}$ is rational if it is deg-local; that is,

$$( - \circ \deg_n ) : (S^1 \to X) \to (S^1 \to X)$$

are equivalences for all $n : \mathbb{N}$.
Rational Types (Christensen, et al '18)

A type $X : \mathcal{U}$ is rational if it is deg-local; that is,

$$(- \circ \text{deg}_n) : (S^1 \rightarrow X) \rightarrow (S^1 \rightarrow X)$$

are equivalences for all $n : \mathbb{N}$.

Rationalization

A rationalization of $X$ is a type $X' : \mathcal{U}$ and a map $g : X \rightarrow X'$ such that for every rational type $Y$, the precomposition map

$$(- \circ g) : (X' \rightarrow Y) \rightarrow (X \rightarrow Y)$$

is an equivalence.
Rational Types (Christensen, et al '18)

A type $X : \mathcal{U}$ is rational if it is deg-local; that is,

$(- \circ \deg_n) : (S^1 \to X) \to (S^1 \to X)$

are equivalences for all $n : \mathbb{N}$.

Rationalization

A rationalization of $X$ is a type $X' : \mathcal{U}$ and a map $g : X \to X'$ such that for every rational type $Y$, the precomposition map

$(- \circ g) : (X' \to Y) \to (X \to Y)$

is an equivalence.

We denote the rationalization of the $n$-sphere by $S^n_Q$, and note that

$\pi_k(S^n_Q) \simeq \pi_k(S^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.
Cohomology of Rational Eilenberg-MacLane Spaces

If $n$ is even, then $H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[e]$, the polynomial algebra over $\mathbb{Q}$ with generator $e : H^n(K(\mathbb{Q}, n))$.

If $n$ is odd, then $H^*(K(\mathbb{Q}, n))$ is concentrated in dimension $n$, i.e. $K(\mathbb{Q}, n)$ is an $n$-(co)homology sphere.
Cohomology of Rational Eilenberg-MacLane Spaces

If $n$ is even, then $H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[e]$, the polynomial algebra over $\mathbb{Q}$ with generator $e : H^n(K(\mathbb{Q}, n))$.

If $n$ is odd, then $H^*(K(\mathbb{Q}, n))$ is concentrated in dimension $n$, i.e. $K(\mathbb{Q}, n)$ is an $n$-(co)homology sphere.

Proof (sketch) by induction:

• $S^1_\mathbb{Q} \simeq K(\mathbb{Q}, 1)$, so the base case is evident.
Cohomology of Rational Eilenberg-MacLane Spaces

If $n$ is even, then $H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[e]$, the polynomial algebra over $\mathbb{Q}$ with generator $e : H^n(K(\mathbb{Q}, n))$.

If $n$ is odd, then $H^*(K(\mathbb{Q}, n))$ is concentrated in dimension $n$, i.e. $K(\mathbb{Q}, n)$ is an $n$-(co)homology sphere.

Proof (sketch) by induction:

• $S^1_\mathbb{Q} \simeq K(\mathbb{Q}, 1)$, so the base case is evident.

• If $n$ is even, then by hypothesis $K(\mathbb{Q}, n - 1)$ is a cohomology sphere and we can apply the Gysin sequence to the path fibration $K(\mathbb{Q}, n - 1) \simeq \Omega K(\mathbb{Q}, n) \to 1 \to K(\mathbb{Q}, n)$. 

Cohomology of Rational Eilenberg-MacLane Spaces

If $n$ is even, then $H^*(K(\mathbb{Q}, n)) = \mathbb{Q}[e]$, the polynomial algebra over $\mathbb{Q}$ with generator $e : H^n(K(\mathbb{Q}, n))$.

If $n$ is odd, then $H^*(K(\mathbb{Q}, n))$ is concentrated in dimension $n$, i.e. $K(\mathbb{Q}, n)$ is an $n$-(co)homology sphere.

Proof (sketch) by induction:

• $S^1_\mathbb{Q} \simeq K(\mathbb{Q}, 1)$, so the base case is evident.

• If $n$ is even, then by hypothesis $K(\mathbb{Q}, n-1)$ is a cohomology sphere and we can apply the Gysin sequence to the path fibration $K(\mathbb{Q}, n-1) \simeq \Omega K(\mathbb{Q}, n) \to 1 \to K(\mathbb{Q}, n)$.

• If $n$ is odd, we analyze the fiber sequence of the truncation $\tau_n : S^n_\mathbb{Q} \to K(\mathbb{Q}, n)$.

$$
\cdots \to K(\mathbb{Q}, n-1) \to \text{fib}_{\tau_n}(\ast_{K(\mathbb{Q}, n)}) \to S^n_\mathbb{Q} \to K(\mathbb{Q}, n)
$$
Sketch of the Finiteness Theorem

For $n$ odd:

- The $n$-truncation $\tau_n: S^n_Q \to K(\mathbb{Q}, n)$ induces an equivalence in cohomology, so its $n$-connected fiber has trivial cohomology.
Sketch of the Finiteness Theorem

For $n$ odd:

- The $n$-truncation $\tau_n : S^n_Q \to K(Q, n)$ induces an equivalence in cohomology, so its $n$-connected fiber has trivial cohomology.
- Use the Postnikov system for the fiber to show that $\text{Hom}(\pi_{n+1}(F), Q) = 0$. 
Sketch of the Finiteness Theorem

For $n$ odd:

- The $n$-truncation $\tau_n : S^n_Q \to K(Q, n)$ induces an equivalence in cohomology, so its $n$-connected fiber has trivial cohomology.
- Use the Postnikov system for the fiber to show that $\text{Hom}(\pi_{n+1}(F), Q) = 0$.
- Appeal to the finite generation of homotopy groups\textsuperscript{†} to conclude that the fiber is in fact $(n + 1)$-connected.
Sketch of the Finiteness Theorem

For $n$ odd:

- The $n$-truncation $\tau_n : S^n_Q \to K(\mathbb{Q}, n)$ induces an equivalence in cohomology, so its $n$-connected fiber has trivial cohomology.
- Use the Postnikov system for the fiber to show that $\text{Hom}(\pi_{n+1}(F), \mathbb{Q}) = 0$.
- Appeal to the finite generation of homotopy groups† to conclude that the fiber is in fact $(n + 1)$-connected.
- By induction, therefore the fiber has trivial homotopy groups.
Sketch of the Finiteness Theorem

For $n$ odd:

- The $n$-truncation $\tau_n : S^n_Q \to K(\mathbb{Q}, n)$ induces an equivalence in cohomology, so its $n$-connected fiber has trivial cohomology.
- Use the Postnikov system for the fiber to show that $\text{Hom}(\pi_{n+1}(F), \mathbb{Q}) = 0$.
- Appeal to the finite generation of homotopy groups\(^\dagger\) to conclude that the fiber is in fact $(n + 1)$-connected.
- By induction, therefore the fiber has trivial homotopy groups.
- Conclude that by the long exact sequence of $\tau_n$, we get that $\pi_* (S^n_Q) \cong \pi_* (K(\mathbb{Q}, n))$ is localized in degree $n$.\(^\ddagger\)
For $n$ even:

- Use the Gysin sequence associated to the fiber sequence
  \[ K(\mathbb{Q}, n - 1) \rightarrow \text{fib}_{\tau_n}(\ast K(\mathbb{Q}, n)) \xrightarrow{j} S^n_Q \] to show that $\text{fib}_{\tau_n}(\ast)$ is a $(2n - 1)$-cohomology sphere.
For $n$ even:

- Use the Gysin sequence associated to the fiber sequence
  \[ K(\mathbb{Q}, n - 1) \to \text{fib}_{\tau_n}(\ast K(\mathbb{Q}, n)) \xrightarrow{j} S^n_Q \]  
  to show that $\text{fib}_{\tau_n}(\ast)$ is a $(2n - 1)$-cohomology sphere.

- Since $\pi_{2n-1}(\text{fib}_{\tau_n}(\ast)) \simeq \pi_{2n-1}(S^n_Q)$ is nontrivial (Brunerie, 2016), there is a nontrivial map $f: S^{2n-1} \to \text{fib}_{\tau_n}(\ast)$. 
For $n$ even:

- Use the Gysin sequence associated to the fiber sequence
  \[ K(\mathbb{Q}, n - 1) \to \text{fib}_{\tau_n}(\ast K(\mathbb{Q}, n)) \xrightarrow{j} S^n_\mathbb{Q} \] to show that \( \text{fib}_{\tau_n}(\ast) \) is a \((2n - 1)\)-cohomology sphere.

- Since \( \pi_{2n-1}(\text{fib}_{\tau_n}(\ast)) \cong \pi_{2n-1}(S^n_\mathbb{Q}) \) is nontrivial (Brunerie, 2016), there is a nontrivial map \( f: S^{2n-1}_\mathbb{Q} \to \text{fib}_{\tau_n}(\ast) \).

- By the universal property of rationalization, since \( \text{fib}_{\tau_n}(\ast) \) is a rational type, there is an induced map \( f_\mathbb{Q}: S^{2n-1}_\mathbb{Q} \to \text{fib}_{\tau_n}(\ast) \).
For $n$ even:

- Use the Gysin sequence associated to the fiber sequence
  \[ K(\mathbb{Q}, n - 1) \to \text{fib}_{\tau_n}(\ast K(\mathbb{Q}, n)) \xrightarrow{j} S^n_{\mathbb{Q}} \] to show that $\text{fib}_{\tau_n}(\ast)$ is a $(2n - 1)$-cohomology sphere.
  
- Since $\pi_{2n-1}(\text{fib}_{\tau_n}(\ast)) \simeq \pi_{2n-1}(S^n_{\mathbb{Q}})$ is nontrivial (Brunerie, 2016), there is a nontrivial map $f : S^{2n-1} \to \text{fib}_{\tau_n}(\ast)$.

- By the universal property of rationalization, since $\text{fib}_{\tau_n}(\ast)$ is a rational type, there is an induced map $f_{\mathbb{Q}} : S^{2n-1}_{\mathbb{Q}} \to \text{fib}_{\tau_n}(\ast)$.

- $f_{\mathbb{Q}}$ induces an isomorphism in cohomology, so therefore also induces an isomorphism in homotopy as well (examine the fiber).
For $n$ even:

- Use the Gysin sequence associated to the fiber sequence
  \[ K(\mathbb{Q}, n - 1) \to \text{fib}_{\tau_n}(\ast K(\mathbb{Q}, n)) \xrightarrow{j} S^n_{\mathbb{Q}} \]
  to show that $\text{fib}_{\tau_n}(\ast)$ is a $(2n - 1)$-cohomology sphere.
- Since $\pi_{2n-1}(\text{fib}_{\tau_n}(\ast)) \simeq \pi_{2n-1}(S^n_{\mathbb{Q}})$ is nontrivial (Brunerie, 2016), there is a nontrivial map $f : S^{2n-1} \to \text{fib}_{\tau_n}(\ast)$.
- By the universal property of rationalization, since $\text{fib}_{\tau_n}(\ast)$ is a rational type, there is an induced map $f_{\mathbb{Q}} : S^{2n-1}_{\mathbb{Q}} \to \text{fib}_{\tau_n}(\ast)$.
- $f_{\mathbb{Q}}$ induces an isomorphism in cohomology, so therefore also induces an isomorphism in homotopy as well (examine the fiber).
- Therefore, for $i \neq n$,
  \[ \pi_i(S^n_{\mathbb{Q}}) \simeq \pi_i(\text{fib}_{\tau_n}) \simeq \pi_i(S^{2n-1}_{\mathbb{Q}}), \]
  so the homotopy groups $\pi_i(S^n_{\mathbb{Q}})$ are localized at $i = n$ and $i = 2n - 1$. 
Reference

Classical References:

- Hatcher, *Algebraic Topology*
- Hatcher, *Vector Bundles & K-Theory*
- May, *A Concise Course in Algebraic Topology*

HoTT:

- Brunerie, *On the Homotopy Groups of Spheres in Homotopy Type Theory*
- Christensen, Opie, Rijke, Scoccola; *Localization in Homotopy Type Theory*
- Favonia, Harper; *Covering Spaces in Homotopy Type Theory*
- Licata, Finster; *Eilenberg-MacLane Spaces in Homotopy Type Theory*