Nominal Sets and the Schanuel Topos

Fernando Larrain

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1 Nominal Sets

1.1 Group Actions

Let G be a group. A *G-set* is a functor over G or, equivalently, a set X equipped with a left action $\mu : G \times X \to X$ of G. An *equivariant map* between G-sets is a natural transformation between them or, equivalently, a function between their underlying sets that respects the action of G.

The category $[G, \mathbf{Set}]$ of G-sets and equivariant maps is an elementary topos because it is a category of presheaves. Furthermore, it is boolean, so it supports classical higher order logic. The subobject classifier is given by the discrete G-set $\mathbb{B} = \{\top, \bot\}$ (a G-set (X, μ) is *discrete* when $\mu = \pi_2$).

The exponentials in $[G, \mathbf{Set}]$ are the exponentials in $\mathbf{Set}$, where the action of $g \in G$ on a function ϕ between G-sets is given by

$$(g \cdot \phi)(x) = g \cdot (\phi(g^{-1} \cdot x)).$$

In the particular case of powersets, this amounts to mapping subsets S of a G-set X to their image $g \cdot S = \{g \cdot x \mid x \in X\}$. If S is closed under the action of G, we say S is *equivariant*. These subsets are important because they correspond precisely to the subobjects of X. Also, quotients of G-sets by equivariant equivalence relations give rise to quotients in $[G, \mathbf{Set}]$ in the obvious way.

1.2 Permutation Groups

Let A be a set. Then, $\operatorname{Sym} A$ denotes the *symmetric group* on A and $\operatorname{Perm} A$, the subgroup of $\operatorname{Sym} A$ of *finite permutations*. A permutation π is said to be *finite* if $\{a \in A \mid \pi(a) \neq a\}$ is.

Given $a, a' \in A$, we define $(a\ a')$ as the finite permutation that swaps a and a' and leaves everything else unchanged. Such a permutation is called a *transposition*. Transpositions generate $\operatorname{Perm} \mathbb{A}$. In fact, for any finite permutation π, we can choose its factors $(a\ a')$ so that

$$\pi(a) \neq a \neq a' \neq \pi(a'),$$

i.e. so that they are neither degenerate nor redundant.

1.3 Nominal Sets

Fix a countably infinite set $\mathbb{A}$. The elements of $\mathbb{A}$ will be called *atomic names*.

Given a $\operatorname{Perm} \mathbb{A}$-set X, we say that $A \subseteq \mathbb{A}$ *supports* $x \in X$ if every permutation that fixes each element in A also fixes x:

$$(\forall a \in A) \pi(a) = a \implies \pi(x) = x.$$

In terms of transpositions, A supports x if

$$(\forall a_1, a_2 \in \mathbb{A} - A)(a_1\ a_2) \cdot x = x.$$

A fundamental property of the set of finite supports of x is that it is closed under binary intersections. In other words, given two finite sets A_1 and A_2, if both support x, then so does $A_1 \cap A_2$. To see this, fix $a_1, a_2 \in \mathbb{A} - (A_1 \cap A_2)$. We want to show that $(a_1\ a_2) \cdot x = x$. Notice that a_1 and a_2 might be in A_1 or in A_2, so we cannot use our hypotheses directly. However, since A_1 and A_2 are finite, we can find an "interpolant" a_3 distinct from a_1 and a_2 that is neither in A_1 nor A_2 to factor the transposition as $(a_1\ a_2) = (a_1\ a_3) \circ (a_2\ a_3) \circ (a_1\ a_3)$ (this presupposes that $a_1 \neq a_2$, but if that is not the case, we are done anyway). That the goal follows from the hypotheses is now evident.

A *nominal set* is a $\mathrm{Perm}\,\mathbb{A}$-set whose every element has finite support. For X a nominal set and $x \in X$, we let $\mathrm{supp}_X(x)$ be the least finite $A \subseteq \mathbb{A}$ supporting x (it exists because of the closure property just mentioned). **Nom** is the full subcategory of $[\mathrm{Perm}\,\mathbb{A}, \mathbf{Set}]$ spanned by the nominal sets. In fact, it is a coreflective subcategory of $[\mathrm{Perm}\,\mathbb{A}, \mathbf{Set}]$, i.e. its associated inclusion functor has a right-adjoint given on $\mathrm{Perm}\,\mathbb{A}$-sets X by

$$X_{\mathrm{fs}} = \{x \in X \mid x \text{ is finitely supported}\}.$$

Furthermore, it is a boolean Grothendieck topos, as we show next. For the proof, we shall require a few basic results about nominal sets.

First, for any finite $A \subseteq \mathbb{A}$, $\mathrm{supp}(A) = A$, so the set $\mathcal{P}_f(\mathbb{A})$ of finite subsets of $\mathbb{A}$ is a nominal set.

Second, for any $\mathrm{Perm}\,\mathbb{A}$-set X, the function $\mathrm{supp}_X : X \to \mathcal{P}_f(\mathbb{A})$ is equivariant, because it is definable in the internal logic of $[\mathrm{Perm}\,\mathbb{A}, \mathbf{Set}]$.

Third, if $f : X \to Y$ is an equivariant function, then it preserves support, i.e. if A supports $x \in X$, then A supports $f(x)$.

1.4 A Category of Contexts and Renamings

Let $\mathbb{I}$ be the category of finite subsets of $\mathbb{A}$ and injections between them, with identities and compositions as in **Set**. We can think of $\mathbb{I}^{\mathrm{op}}$ as a category of *(naming) contexts* and *renamings* (contractions excluded).

Proposition 1. *There is a functor* $I_* : \mathbf{Nom} \to [\mathbb{I}, \mathbf{Set}]$.

Proof. Every nominal set X gives rise to a presheaf F on $\mathbb{I}^{\mathrm{op}}$:

1. On contexts $\Gamma \in \mathbb{I}$, we put $F(\Gamma) = \{x \in X \mid \mathrm{supp}(x) \subseteq \Gamma\}$.

2. On renamings $\Gamma \xrightarrow{\rho} \Delta$ in $\mathbb{I}$, we let $F(\rho)(x) = \pi \cdot x$, where $\pi \in \mathrm{Perm}\,\mathbb{A}$ satisfies $\pi \restriction \Gamma = \rho$.

Notice that every injection $\rho : \Gamma \to \Delta$ in $\mathbb{I}$ can be extended to a finite permutation π on $\mathbb{A}$, so that $F(\rho)$ is always defined. Indeed, since $\Gamma \cong \operatorname{im}\rho$,

$$\begin{aligned} |\operatorname{im}\rho - \Gamma| &= |\operatorname{im}\rho| - |\Gamma \cap \operatorname{im}\rho|, \\ &= |\Gamma| - |\Gamma \cap \operatorname{im}\rho|, \\ &= |\operatorname{im}\rho - \Gamma|, \end{aligned}$$

so there is a bijection $\rho' : (\operatorname{im}\rho - \Gamma) \to (\Gamma - \operatorname{im}\rho)$ and one can set

$$\pi(a) = \begin{cases} \rho(a) & a \in \Gamma, \\ \rho'(a) & a \in \operatorname{im}\rho - \Gamma, \\ a & \text{otherwise} \end{cases}$$

Furthermore, the behavior of $F(\rho)$ does not depend on the choice of extension π. Indeed, if π and π' extend ρ, then $\pi^{-1} \circ \pi'$ fixes each element of Γ, which supports every $x \in F(\Gamma)$, so that $(\pi^{-1} \circ \pi') \cdot x = x$ and hence $\pi' \cdot x = \pi \cdot x$.

Lastly, $\operatorname{supp}(F(\rho)(x)) = \operatorname{supp}(\pi \cdot x) = \pi \cdot \operatorname{supp}(x) \subseteq \pi \cdot \Gamma \subseteq \Delta$, so $F(\rho)(x) \in F(\Delta)$.

Since the choice of extension is irrelevant, it is easy to see that F is a functor. Thus, we obtain a function $I_* : \mathbf{Nom} \to [\mathbb{I}, \mathbf{Set}]$. This function extends to a functor as follows. Consider an equivariant function $f : X \to Y$. Define a natural transformation $\eta : I_*(X) \to I_*(Y)$ by setting $\eta_\Gamma = f \restriction I_*(X)(\Gamma)$. This is well defined because, for any $x \in I_*(X)(\Gamma)$,

$$\operatorname{supp}(f(x)) \subseteq \operatorname{supp}(x) \subseteq \Gamma.$$

Naturality follows from the equivariance of f. □

Proposition 2. *The functor $I_* : \mathbf{Nom} \to [\mathbb{I}, \mathbf{Set}]$ is full and faithful.*

Proof.
1. I_* is faithful: This is essentially due to the fact that, for any $X \in \mathbf{Nom}$, every $x \in X$ is in some fiber of $I_*(X)$. Indeed, let $f, f' : X \rightrightarrows Y$ be equivariant maps such that $f = f'$. Then, for each $x \in X$,

$$f(x) = I_*(f)_{\operatorname{supp}(x)}(x) = I_*(f')_{\operatorname{supp}(x)}(x) = f'(x).$$

2. I_* is full: Fix $\alpha : I_*(X) \to I_*(Y)$ in $[\mathbb{I}, \mathbf{Set}]$. Let $f : X \to Y$ be the function $x \mapsto \alpha_{\operatorname{supp}(x)}(x)$. To see that it is equivariant, fix a finite permutation π and a point $x \in X$. Notice that π restricts to an injection $\rho : \operatorname{supp}(x) \to \operatorname{supp}(\pi \cdot x)$, since $\pi \cdot \operatorname{supp}(x) = \operatorname{supp}(\pi \cdot x)$. The associated naturality square implies that $f(\pi \cdot x) = \pi \cdot f(x)$. It remains to show that $I_*(f) = \alpha$, so fix $\Gamma \in \mathbb{I}$ and $x \in I_*(X)(\Gamma)$ and let

$\iota : \mathrm{supp}(x) \to \Gamma$ be the obvious inclusion. Then,

$$\begin{aligned} I_*(f)(x) &= f(x), \\ &= \alpha_{\mathrm{supp}(x)}(x), \\ &= id_{\mathbb{A}} \cdot (\alpha_{\mathrm{supp}(x)}(x)), \\ &= I_*(Y)(\iota)(\alpha_{\mathrm{supp}(x)}(x)), \\ &= \alpha_\Gamma(I_*(X)(\iota)(x)), \\ &= \alpha_\Gamma(id_{\mathbb{A}} \cdot x), \\ &= \alpha_\Gamma(x). \end{aligned}$$

□

2 The Schanuel Topos

2.1 A Site of Contexts and Renamings

Let $F : \mathbb{I} \to \mathbf{Set}$, $\Gamma \in \mathbb{I}$ and $x \in F(\Gamma)$. We say that a subcontext $\Delta \xrightarrow{\rho} \Gamma$ of Γ *supports* x whenever renamings of Γ that agree on Δ act equally on x:

$$\rho_1 \circ \rho = \rho_2 \circ \rho \implies \rho_1 \cdot x = \rho_2 \cdot x$$

for every Υ and $\rho_1, \rho_2 : \Gamma \rightrightarrows \Upsilon$ in $\mathbb{I}$. For example, ρ supports every x in the image of $F(\rho)$, since then $\rho_1 \cdot x = \rho_1 \cdot \rho \cdot y = \rho_2 \cdot \rho \cdot y = \rho_2 \cdot x$ for some $y \in F(\Delta)$ whenever $\rho_1 \circ \rho = \rho_2 \circ \rho$.

If $\Delta \xrightarrow{\rho} \Gamma$ supports x, we would like to say that x lies in the fiber of Δ (modulo permutation of names). In other words, we would like there to be a canonical way of strengthening x to Δ along ρ. The presheaves on $\mathbb{I}^{\mathrm{op}}$ for which this is always possible are the sheaves for the atomic topology on $\mathbb{I}^{\mathrm{op}}$, and the category of all such sheaves is called the *Schanuel Topos*.

Recall that the *atomic topology* on $\mathbb{I}^{\mathrm{op}}$ is given by the family of all nonempty sieves on objects of $\mathbb{I}^{\mathrm{op}}$. This family is a Grothendieck topology if, and only if, we can complete any cospan in $\mathbb{I}^{\mathrm{op}}$ to a commutative square or, equivalently, if we can complete any span in $\mathbb{I}$ to a commutative square, but this is indeed the case:

$$\begin{array}{ccc} \Upsilon & \xrightarrow{\quad\sigma\quad} & \Gamma \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\iota_2} \\ \Delta & \xrightarrow[\iota_1]{} & (\Delta - \mathrm{im}\,\rho) + \Upsilon + (\Gamma - \mathrm{im}\,\sigma) \end{array}$$

where $\iota_1 = id_{\Delta - \mathrm{im}\,\rho} + \rho^{-1}$ and $i_2 = id_{\Gamma - \mathrm{im}\,\sigma} + \sigma^{-1}$.

We say that every morphism in $\mathbb{I}^{\mathrm{op}}$ covers because the atomic topology on $\mathbb{I}^{\mathrm{op}}$ is generated by the singleton families of morphisms. To see this, consider a nonemtpy sieve S on $\Delta \in \mathbb{I}^{\mathrm{op}}$. Since it is nonempty, it contains a map $\Delta \xrightarrow{\rho} \Gamma$ in $\mathbb{I}$ such that

$$|\Gamma| = \min\{|\Theta| : \Theta \in \mathbb{I}, S(\Theta) \neq \emptyset\}.$$

Consider any $\Theta \in \mathbb{I}$ and $\sigma \in S(\Theta)$. Then,

$$\begin{aligned} |\Gamma - \operatorname{im}\rho| &= |\Gamma| - |\operatorname{im}\rho|, \\ & |\Gamma| - |\Delta|, \\ &\leq |\Theta| - |\Delta|, \\ &= |\Theta| - |\operatorname{im}\sigma|, \\ &= |\Theta - \operatorname{im}\sigma|, \end{aligned}$$

so σ factors through ρ and, consequently, σ is in the sieve generated by ρ. Conversely, if $\sigma : \Delta \to \Theta$ is a map in $\mathbb{I}$ that factors through ρ, then $\sigma = \mathsf{y}(\tau)(\rho)$ for some $\tau : \Gamma \to \Theta$ in I. Since $\rho \in S(\Gamma)$ and S is a subpresheaf of $\mathsf{y}(\Delta)$, it follows that $\sigma \in S(\Theta)$.

Notice that the notion of support corresponds to that of *matching family* for the atomic topology. More precisely, for each $F : \mathbb{I} \to \mathbf{Set}$ and $\Delta \xrightarrow{\rho} \Gamma$ in $\mathbb{I}$, we have a bijection

$$\mathrm{Nat}(S, F) \cong \{x \in F(\Gamma) \mid \rho \text{ supports } x\},$$

where S is the sieve generated by ρ, as in Yoneda's Lemma. Explicitly, if $x \in \Gamma$ is supported by ρ, then we let

$$\phi_\Theta(\sigma) = \tau \cdot x,$$

where $\tau : \Delta \to \Theta$ is such that $\tau \circ \rho = \sigma$. Notice that τ exists by definition of S, and the meaning of $\phi_\Theta(\sigma)$ is independent of τ precisely because ρ supports x.

Conversely, if $\phi : S \to F$, then we let $x = \phi_\Gamma(\rho)$, which is supported by ρ because ϕ is natural. Indeed, for any $\tau : \Gamma \to \Theta$, naturality implies that $\tau \cdot x = \phi_\Theta(\tau \circ \rho)$, so that $\tau \circ \rho$ determines x.

These constructions are mutual inverses. Indeed, let $x \in F(\Gamma)$ supported by ρ and let ϕ be the matching family corresponding to x. Then, $\phi_\Gamma(\rho) = id_\Gamma \cdot x = x$. Now let $x = \psi_\Gamma(\rho)$ for some matching family ψ. Then, $\phi_\Theta(\sigma) = \tau \cdot x = \tau \cdot \psi_\Gamma(\rho) = \psi_\Theta(\tau \circ \rho) = \psi_\Theta(\sigma)$, where $\sigma = \tau \circ \rho$.

2.2 The Schanuel Topos

We have defined the Schanuel Topos as the category of sheaves on $\mathbb{I}^{\mathrm{op}}$ for the atomic topology. Next, we characterize its objects in more detail.

Fix a functor $F : \mathbb{I} \to \mathbf{Set}$. Notice, first, that the *amalgamations* of a matching family $\phi : S \to F$ on the sieve S generated by $\Delta \xrightarrow{\rho} \Gamma$ correspond precisely to the preimages of $x = \phi_\Gamma(\rho)$ under the action of ρ. To see this, let ψ be an amalgamation of ϕ and $y = \psi_\Delta(id_\Delta)$. Then, $\rho \cdot y = \psi_\Gamma(\rho \circ id_\Delta) = \phi_\Gamma(\rho) = x$. Conversely, let $\rho \cdot y = x$ and $\psi = (-) \cdot y$. Then, $\psi_\Theta(\sigma) = \sigma \cdot y = \tau \cdot \rho \cdot y = \tau \cdot x = \phi_\Theta(\sigma)$, where $\sigma = \tau \circ \rho$. These two mappings are inverses. Indeed, $\psi_\Delta(id_\Delta) = id_\Delta \cdot y$. In the other direction, $(-) \cdot \psi_\Delta(id_\Delta) = \psi$, since $\sigma \cdot \psi_\Delta(id_\Delta) = \psi_\Theta(\sigma \circ id_\Delta)$ for every $\sigma : \Delta \to \Theta$.

Now, recall that F satisfies the separation condition for S if, and only if, every matching family on S has at most one amalgamation. In particular, $F(\rho)$ must be injective (recall that ρ supports everything in the image of $F(\rho)$). Recall also that F satisfies the sheaf condition for S if, and only if, every matching family on S has a unique amalgamation. In particular, the points supported by ρ must lie in the image of $F(\rho)$. But if that is the case, injectivity is not only necessary but sufficient for separation. Consequently, F satisfies the sheaf condition for S if, and only if, $F(\rho)$ is injective and its image contains every point supported by ρ. In other words,

Proposition 3. $F : \mathbb{I} \to \mathbf{Set}$ *is a sheaf for the atomic topology on* $\mathbb{I}^{\mathrm{op}}$ *if, and only if, whenever* $\Delta \xrightarrow{\rho} \Gamma$ *in* $\mathbb{I}$ *supports* $x \in F(\Gamma)$*, there is a unique* $y \in F(\Delta)$ *such that* $\rho \cdot y = x$.

Next, we provide another characterization of sheaves that will come in handy when describing the essential image of $I_* : \mathbf{Nom} \to [\mathbb{I}, \mathbf{Set}]$. It corresponds to the fact that the finite supports of a point in a Perm $\mathbb{A}$-set are closed under binary intersection.

Proposition 4. $F : \mathbb{I} \to \mathbf{Set}$ *is a sheaf for the atomic topology on* $\mathbb{I}^{\mathrm{op}}$ *if, and only if,* F *is pullback-preserving.*

Proof. In the proof, we shall make use of the fact that the inclusion of $\mathbb{I}$ into $\mathbf{Set}$ creates pullbacks.

Suppose that F is pullback-preserving. Suppose, furthermore, that $\Delta \xrightarrow{\rho} \Gamma$ in $\mathbb{I}$ supports $x \in F(\Gamma)$. Consider the pullback square

$$\begin{array}{ccc} \Delta & \xrightarrow{\quad\rho\quad} & \Gamma \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\iota_2} \\ \Gamma & \xrightarrow[\iota_1]{} & (\Gamma - \mathrm{im}\,\rho) + \Delta + (\Gamma - \mathrm{im}\,\rho) \end{array}$$

where ι_1 and ι_2 are defined as before. Since F preserves pullbacks and ρ supports x, there is a unique $y \in F(\Delta)$ such that $\rho \cdot y = x$. Hence, F is a sheaf.

Conversely, suppose that F is a sheaf. Fix a pullback square

$$\begin{array}{ccc} \Delta & \xrightarrow{\rho} & \Gamma \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ \Theta & \xrightarrow[\xi]{} & \Upsilon \end{array}$$

Fix also $x \in F(\Gamma)$ and $y \in F(\Theta)$ such that $\tau \cdot x = \xi \cdot y$. We shall show that ρ supports x, so that there is a unique $z \in F(\Delta)$ such that $\rho \cdot z = x$. Notice that such a z also satisfies $\sigma \cdot z = y$, since $\xi \cdot \sigma \cdot z = \tau \cdot \rho \cdot z = \tau \cdot z = \xi \cdot y$ and $F(\xi)$ is injective.

Now fix two maps $\alpha, \beta : \Gamma \rightrightarrows \Gamma'$ such that $\alpha \circ \rho = \beta \circ \rho$. We claim that $\alpha \cdot x = \beta \cdot x$. To see this, notice that we can extend the previous diagram as follows:

$$\begin{array}{ccccc} \Delta & \xrightarrow{\rho} & \Gamma & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & \Xi \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\iota} \\ \Theta & \xrightarrow[\xi]{} & \Upsilon & \overset{\gamma}{\underset{\delta}{\rightrightarrows}} & (\Upsilon - \operatorname{im}\tau) + \Xi \end{array}$$

where $\iota \circ \alpha = \gamma \circ \tau$, $\iota \circ \beta = \gamma \circ \tau$ and $\gamma \circ \xi = \delta \circ \xi$. Here, ι is the coproduct inclusion of Ξ and γ and δ are defined as follows:

$$\gamma(a) = \begin{cases} \alpha(\tau^{-1}(a)) & a \in \operatorname{im}\tau \\ a & \text{otherwise} \end{cases}$$

$$\delta(a) = \begin{cases} \beta(\tau^{-1}(a)) & a \in \operatorname{im}\tau \\ a & \text{otherwise} \end{cases}$$

Then, $\iota \cdot \alpha \cdot x = \gamma \cdot \tau \cdot x = \gamma \cdot \xi \cdot y = \delta \cdot \xi \cdot y = \delta \cdot \tau \cdot x = \iota \cdot \beta \cdot x$. Since $F(\iota)$ is injective, $\alpha \cdot x = \beta \cdot x$, as needed. $\square$

2.3 The Left-Adjoint

Proposition 5. *The inclusion* $I_* : \mathbf{Nom} \hookrightarrow [\mathbb{I}, \mathbf{Set}]$ *has a left-adjoint, which we denote by* $I^* : [\mathbb{I}, \mathbf{Set}] \to \mathbf{Nom}$.

Proof. Every presheaf F on $\mathbb{I}^{\mathrm{op}}$ gives rise to a nominal set X:

1. The wide subcategory of $\mathbb{I}$ consisting of the canonical inclusions is a directed set. Take X to be the colimit of the restriction of F to this directed set. Because it is directed, the colimit can be explicitly described as the quotient of $\sum_{\Gamma \in \mathbb{A}} F(\Gamma)$ by the equivalence relation

$$(\Gamma, x) \sim (\Delta, y) \iff (\exists \Upsilon \supseteq \Gamma \cup \Delta) F(\Gamma \hookrightarrow \Upsilon)(x) = F(\Delta \hookrightarrow \Upsilon)(y).$$

   We will write $[\Gamma, x]$ for the equivalence class of (Γ, x).

2. Take the action of $\pi \in \operatorname{Perm}\mathbb{A}$ on $[\Gamma, x]$ to be

$$\pi \cdot [\Gamma, x] = [\pi \cdot \Gamma, F(\pi \restriction \Gamma)(x)].$$

This action is well defined. Indeed, suppose that $(\Gamma, x) \sim (\Delta, y)$. Let $\Upsilon \supseteq \Gamma \cup \Delta$ such that $F(\Gamma \hookrightarrow \Upsilon)(x) = F(\Delta \hookrightarrow \Upsilon)(y)$. Then, we have that

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi\restriction\Gamma} & \pi\cdot\Gamma \\ \big\downarrow & & \big\downarrow \\ \Upsilon & \xrightarrow[\pi\restriction\Upsilon]{} & \pi\cdot\Upsilon \end{array}$$

and similarly for Δ, so that $(\pi \cdot \Gamma, F(\pi \restriction \Gamma)(x)) \sim (\pi \cdot \Delta, F(\pi \restriction \Delta)(x))$.

3. Let $[\Gamma, x] \in X$ and suppose $(\forall a \in \Gamma)\pi(a) = a$. Then, $\pi \cdot A = A$ and $\pi \restriction \Gamma = id_\Gamma$, so $\pi \cdot [\Gamma, x] = [\Gamma, x]$. Hence, Γ supports $[\Gamma, x]$.

It is not hard to show that the family of maps $x \in F(\Gamma) \mapsto [\Gamma, x]$ indexed by $\Gamma \in \mathbb{I}$ gives rise to a natural transformation $\eta_F : F \to I_*(X)$. We claim that, for any nominal set Y and natural transformation $\alpha : F \to I_*(Y)$, there is a unique equivariant function $\hat{\alpha} : X \to Y$ such that $\alpha = I_*(\hat{\alpha}) \circ \eta_F$:

$$\begin{array}{ccc} F & & \\ {\scriptstyle \eta_F}\big\downarrow & \searrow{\scriptstyle \alpha} & \\ I_*(X) & \dashrightarrow & I_*(Y) \end{array}$$

$$X \overset{}{\underset{\hat{\alpha}}{\dashrightarrow}} Y$$

For uniqueness, notice that, if such a $\hat{\alpha}$ existed, then we'd have

$$\begin{aligned} \hat{\alpha}[\Gamma, x] &= (I_*(\hat{\alpha}))_\Gamma((\eta_F)_\Gamma(x)), \\ &= \alpha_\Gamma(x) \end{aligned}$$

For existence, notice that such an assignment is well defined, for if $(\Gamma, x) \sim (\Delta, y)$, then, for some $\Upsilon \supseteq \Gamma \cup \Delta$,

$$\begin{aligned} \alpha_\Gamma(x) &= id_{\mathbb{A}} \cdot \alpha_\Gamma(x), \\ &= I_*(Y)(\Gamma \hookrightarrow \Upsilon)(\alpha_\Gamma(x)), \\ &= \alpha_\Upsilon(F(\Gamma \hookrightarrow \Upsilon)(x)), \\ &= \alpha_\Upsilon(F(\Delta \hookrightarrow \Upsilon)(y)), \\ &= I_*(Y)(\Delta \hookrightarrow \Upsilon)(\alpha_\Delta(y)), \\ &= id_{\mathbb{A}} \cdot \alpha_\Delta(x), \\ &= \alpha_\Delta(x). \end{aligned}$$

Notice, furthermore, that it is equivariant, since for any $\pi \in \mathrm{Perm}\,\mathbb{A}$,

$$\begin{aligned}\pi \cdot \alpha_\Gamma(x) &= I_*(Y)(\pi \restriction \Gamma)(\alpha_\Gamma(x)),\\ &= \alpha_{\pi\cdot\Gamma}(F(\pi \restriction)(x)).\end{aligned}$$

□

Proposition 6. **Nom** *is equivalent to the Schanuel topos.*

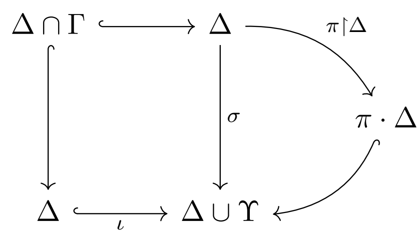
Proof. It suffices to show that the Schanuel topos is the essential image of $I_* : \mathbf{Nom} \hookrightarrow [\mathbb{I}, \mathbf{Set}]$ which, as we have seen, is full and faithful. So fix a sheaf F. Let $X = I^*(F)$. We claim that η_F is an isomorphism. Fix $\Gamma \in \mathbb{I}$.

Notice, first, that $(\eta_F)_\Gamma$ is injective. Indeed, if $(\Gamma, x) \sim (\Gamma, x')$ for some $x, x' \in F(\Gamma)$, then $F(\Gamma \hookrightarrow \Upsilon)(x) = F(\Gamma \hookrightarrow \Upsilon)(x')$ for some $\Upsilon \supseteq \Gamma$; since F is a sheaf, it follows that $x = x'$.

Next, we show that $(\eta_F)_\Gamma$ is surjective. Suppose $[\Delta, y]$ is supported by Γ. We also know that $[\Delta, y]$ is supported by Δ. Hence, $[\Delta, y]$ is supported by $\Delta \cap \Gamma$. Let $\rho : (\Delta - \Gamma) \xrightarrow{\cong} \Upsilon$ with Υ disjoint from Δ. Define $\sigma : \Delta \to \Delta \cup \Upsilon$ as follows:

$$\sigma(a) = \begin{cases} a & a \in \Gamma, \\ \rho(a) & a \notin \Gamma. \end{cases}$$

and let π be a finite permutation extending σ. Then, we obtain the following pullback square:



It's image under F is, therefore, a pullback square. Notice that

$$\begin{aligned}[\Delta \cup \Upsilon, F(\iota)(y)] &= [\Delta, y], && \text{by definition,}\\ &= \pi \cdot [\Delta, y], && \pi \text{ fixes } \Delta \cap \Gamma \text{ pointwise,}\\ &= [\pi \cdot \Delta, F(\pi \restriction \Delta)(y)], && \text{by definition,}\\ &= [\Delta, F(\sigma)(y)], && \text{by definition.}\end{aligned}$$

Since $(\eta_F)_\Gamma$ is injective, $F(\iota)(y) = F(\sigma)(y)$ and, consequently, there is a unique $z \in F(\Delta \cap \Gamma)$ such that $F(\Delta \cap \Gamma \hookrightarrow \Delta)(z) = y$. Now let $x = F(\Delta \cap \Gamma \hookrightarrow \Gamma)(z)$. Then,

$$\begin{aligned}F(\Gamma \hookrightarrow \Delta \cup \Gamma)(x) &= F(\Gamma \hookrightarrow \Delta \cup \Gamma)(F(\Delta \cap \Gamma \hookrightarrow \Gamma)(z)),\\ &= F(\Delta \hookrightarrow \Delta \cup \Gamma)(F(\Delta \cap \Gamma \hookrightarrow \Delta)(z)),\\ &= F(\Delta \hookrightarrow \Delta \cup \Gamma)(y),\end{aligned}$$

so $[\Gamma, x] = [\Delta, y]$ indeed. □

3 References

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