Topos Theory Seminar, Carnegie Mellon University

Giraud's Theorem

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$$\mathscr{T} \underset{\hookrightarrow}{\overset{\text{lex}}{\longleftarrow}} \mathbf{PSh}(\mathbf{C})$$

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Outline

Theoretical Background: Basic Definitions

Cocompleteness

A category is called **Cocomplete** if every diagram in it has a colimit.

Presheaves

Given small category $\mathbf{C}$ we call a functor $F : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$ a **Presheaf**. Replace Set by any category S and it becomes an S-valued presheaf. The category of presheaves on $\mathbf{C}$ is denoted $\hat{\mathbf{C}}$ or $\mathrm{PSh}(\mathbf{C})$.

Finite Presentability

Let $C \in \mathrm{obj}(\mathbf{C})$, then we say c is **Finitely Presentable** if its corresponding Hom-functor, $\mathrm{Hom}(C, -) : \mathbf{C} \to \mathbf{Set}$, preserves (commutes with) directed colimits.

Theoretical Background: λ-Presentability

Definition

Let λ be a regular cardinal.

- A poset P is **λ-Directed** when every $S \subseteq P, |S| < \lambda$ has a join (upper bound). A diagram whose set of morphisms is a λ-directed poset is a λ-directed diagram.
- An object $C \in \text{obj}(\mathbf{C})$ is **λ-Presentable** when $\text{Hom}(C, -)$ preserves λ-directed colimits.

Definition

A category $\mathbf{C}$ is **Locally λ-Presentable** if it is cocomplete and has a set $\mathscr{A} \subseteq \text{obj}(\mathbf{C})$ of λ-presentable objects in that every $C \in \text{obj}(\mathbf{C})$ is a λ-directed colimit of elements in $\mathscr{A}$.

$\mathbf{C}$ is **Locally Presentable** or simply Presentable if it is locally λ-presentable for some λ.

The Goal

Throughout, “topos” will always refer to a Grothendieck topos.

Theorem (Giraud)

Let $\mathscr{X}$ be any category. Then the following conditions are equivalent:

1. *The category $\mathscr{X}$ is a topos; that is, equivalent to the category of sheaves on a site.*
2. *The category $\mathscr{X}$ is a left exact localization of $\mathrm{PSh}(\boldsymbol{C})$ for some small category $\boldsymbol{C}$.*
3. *Giraud's Axioms are satisfied. (To be stated.)*

In this talk we will recall the direction ❶ ⇒ ❷ which is classical from sheafification; then prove ❷ ⇒ ❸ and conclude with the main point of showing ❸ ⇒ ❶.

Size Issues

Throughout we will consider throughout our category **C** to be a small category in the sense that its hom set is indeed a set (as opposed to a proper class) and similarly with our set of objects.

Roughly, a set is just that, and a proper class is “too big”. Rigorously, we have:

Set vs. Proper Class

Given a Grothendieck universe $\mathcal{U}$, we say the **Sets** are elements of $\mathcal{U}$, while **Proper Classes** are *subsets* of $\mathcal{U}$. Thus, an item is small when it is an element of $\mathcal{U}$.

Grothendieck Topologies

Grothendieck Topology

A Grothendieck Topology on category $\mathbf{C}$ is a function J which assigns to each $C \in \mathrm{obj}(\mathbf{C})$ a collection $J(C)$ of sieves on C such that:

- The maximal sieve $\{f \mid \mathrm{cod}(f) = C\}$ is in $J(C)$;
- (Stability) if $S \in J(C)$ then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;
- (Transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D), \forall h : D \to C$ in S then $R \in J(C)$.

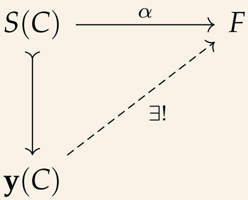
Sites

A **Site** $(\mathbf{C}, J)$ is a category equipped a Grothendieck topology.

Definition of a Topos

Sheaves & Topoi

A presheaf F is a **Sheaf** when for all covering sieves S and all natural transformations $\alpha : S \to F$ there exists a unique extension to the representable functor of $\mathbf{C}$. That is, we have the following diagram:



We call category $\mathscr{X}$ a **(Grothendieck) Topos** if it can be realized as a category of sheaves on some Grothendieck site $\mathbf{C}$, which may be written as $\mathrm{Shv}(\mathbf{C})$.

Localizations

Definition

We call a functor $F : \mathbf{C} \to \mathbf{D}$ a **Localization** if it admits a fully faithful right adjoint. If the left adjoint preserves small limits then F is **Exact**.

Proposition

The category $\mathrm{Shv}(\boldsymbol{C})$ *is a left exact localization of* $\mathrm{PSh}(\boldsymbol{C})$.

Note the above proposition is ❶ $\Rightarrow$ ❷ in Giraud's Theorem.

Effective Epimorphism

Given $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ we say it is an **Effective Epimorphism** if Y is the coimage of f. Equivalently, for the kernel pair $X \times_Y X$, we have $X \times_Y X \rightrightarrows X \xrightarrow{f} Y$ as a coequalizer. Alternatively, $\forall C \in \mathrm{obj}(\mathbf{C})$, $\mathrm{Hom}_{\mathbf{C}}(Y, C) \cong \{u \in \mathrm{Hom}_{\mathbf{C}}(X, C) \mid u \circ \pi = u \circ \pi'\}$ for $\pi, \pi' : X \times_Y X \to X$.

Sheafification

The discussion herein briefly sketches a proof of the preceding proposition and thus a step in our overall proof. See MacLane and Moerdijk 2012 for full details.

Begin with the inclusion functor $i : \mathrm{Shv}(\mathbf{C}) \hookrightarrow \mathrm{PSh}(\mathbf{C})$ then the claim is it has left adjoint $\mathfrak{a} : \mathrm{PSh}(\mathbf{C}) \to \mathrm{Shv}(\mathbf{C})$ where $\mathfrak{a} = \eta \circ \eta$ for $\eta : \mathrm{PSh}(\mathbf{C}) \to \mathrm{PSh}(\mathbf{C})$ such that, for any presheaf F:

$$\eta(F)(C) = \operatorname*{colim}_{R \in J(c)} \mathrm{Match}(R, F).$$

For matching families of the cover R of $C \in \mathrm{obj}(\mathbf{C})$.

Under this construction $\eta(F)$ is a separated presheaf and we may invoke the lemma that any such separated presheaf is a sheaf. Finally, it is easy enough to show that η preserves small limits. (As $\mathrm{Hom}(R, -)$ preserves limits, filtered colimits commute with finite limits in Set and limits in $\mathrm{PSh}(\mathbf{C})$ are computed point-wise.)

Characterizing Grothendieck Topoi

Proposition

Let $\mathscr{X}$ be a category, then the following are equivalent:

1. *The category $\mathscr{X}$ is a Grothendieck topos, that is, equivalent to the category of sheaves on a site.*
2. *The category $\mathscr{X}$ is a lex localization of presheaves for some small category.*

Giraud's Theorem (Again)

Theorem (Giraud)

Let $\mathscr{X}$ be any category. Then the following conditions are equivalent:

1. *The category $\mathscr{X}$ is a topos.*
2. *The category $\mathscr{X}$ is a left exact localization of* $\mathrm{PSh}(\boldsymbol{C})$ *for some small category* $\boldsymbol{C}$*.*
3. *Giraud's Axioms are satisfied:*
   - i. *The category $\mathscr{X}$ is presentable.*
   - ii. *Colimits in $\mathscr{X}$ are universal.*
   - iii. *Coproducts in $\mathscr{X}$ are disjoint.*
   - iv. *Equivalence relations in $\mathscr{X}$ are effective.*

Thus, our next goal is to show that a lex localization of presheaves has exactly these four properties.

[(2) ⇒ (3)] (i)

Proposition (i)

Let $L : \mathrm{PSh}(\boldsymbol{C}) \to \mathscr{X}$ *be a lex localization, then* $\mathscr{X}$ *is presentable.*

See Borceux 1994 for a proof in several steps or Adamek and Rosicky 1994 Theorem 2.26.

Definition

Let $\mathscr{X}$ be a category with pullbacks and small colimits. Then given any morphism $f : T \to S$ we have the adjunction

$$\mathscr{X}_{/S} \underset{f^*}{\overset{f\circ -}{\underset{\perp}{\rightleftarrows}}} \mathscr{X}_{/T}.$$

We say **Colimits in $\mathscr{X}$ are Universal** when the pullback functor $f^* : \mathscr{X}_{/S} \to \mathscr{X}_{/T}$ preserves colimits.

Proposition (ii)

Let $\mathrm{PSh}(\boldsymbol{C}) \xrightarrow{L} \mathscr{X}$ *be a lex localization, then colimits in* $\mathscr{X}$ *are universal.*

Proof of Proposition (ii)

Proof.

Note colimits in $\mathrm{PSh}(\mathbf{C})$ are universal (as they're computed point-wise in Set). Now apply the right adjoint of L to f^* and recall that lex localizations are stable under slice constructions so we get that $f^* : \mathrm{PSh}(\mathbf{C})_{/Y} \to \mathrm{PSh}(\mathbf{C})_{/X}$ preserves colimits. Thus, if we apply L to f^* we conclude $f^* : \mathscr{X}_{/Y} \to \mathscr{X}_{/X}$ preserves colimits. □

$[(2) \Rightarrow (3)]$ (iii)

Definition

Let **C** be a category with coproducts and an initial object $\emptyset$. The coproducts in **C** are **Disjoint** if we have the following pullback diagram:

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \rightarrowtail & X \coprod Y \end{array}$$

and we have that $X \times_{X \coprod Y} Y$ is the initial object in **C**.

Proposition (iii)

Let $L : \mathrm{PSh}(\boldsymbol{C}) \to \mathscr{X}$ *be a lex localization, then coproducts in* $\mathscr{X}$ *are disjoint.*

Proof of Proposition (iii)

Proof.

Let $X, Y \in \mathrm{obj}(\mathscr{X})$ and apply the right adjoint R of L to obtain (using the fact that coproducts in $\mathrm{PSh}(\mathbf{C})$ are disjoint (because again they are computed point-wise in Set and in set and coprods in Set are just disjoint unions) the diagram:

$$\begin{array}{ccc} 0 & \longrightarrow & R(Y) \\ \big\downarrow & \lrcorner & \big\downarrow \\ R(X) & \longrightarrow & RX \coprod RY \end{array}$$

Now we apply L and obtain:

$$\begin{array}{ccc} L(0) & \longrightarrow & LR(Y) \\ \big\downarrow & \lrcorner & \big\downarrow \\ LR(X) & \longrightarrow & L(RX \coprod RY) \end{array}$$

Proof of Proposition (iii) [Cont.]

Proof.

So by left exactness we obtain finally:

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \rightarrowtail & X \coprod Y \end{array}$$

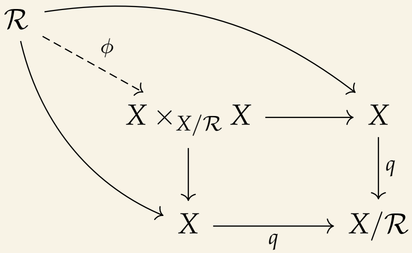
And we see that the coproducts are indeed disjoint.

□

$[(2) \Rightarrow (3)]$ (iv)

Definition

Given equivalence relation $\mathcal{R}$ for a category $\mathscr{X}$ with diagram:



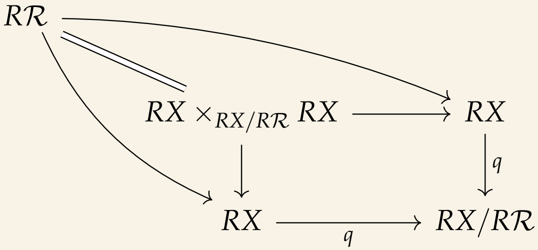
if ϕ is an equality then $\mathcal{R}$ is an **Effective Equivalence Relation**.

Proposition (iv)

Let $L : \mathrm{PSh}(\boldsymbol{C}) \to \mathscr{X}$ be a lex localization, then equivalence relations in $\mathscr{X}$ are effective.

Proof.

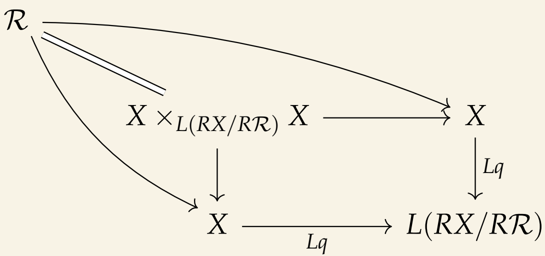
Note first that equivalence relations in Set are effective because we always have pullbacks in Set. Second, it follows that equivalence relations in $\mathrm{PSh}(\mathbf{C})$ are effective . Let $\mathcal{R}$ be an equivalence relation, then we apply the right adjoint R of L and obtain the following diagram in presheaves:



Proof of Proposition (iv) [Cont.]

Proof.

We now apply L (which as a left adjoint preserves colimits and as a lex functor preserves pullbacks) and obtain:



From which we conclude the equivalence relations in $\mathscr{X}$ are effective. □

A Covering Definition

Definition

Let $\mathscr{X}$ be a pretopos which admits infinite coproducts. A collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a **Covering** if it induces an effective epimorphism $\coprod U_i \to X$.

This is equivalent to requiring that for every subobject $X_0 \subseteq X$ for which f_i factors through X_0 we have $X_0 = X$.

Giraud's Theorem (Unpacked)

Theorem (Giraud)

Let $\mathscr{X}$ be any category. Then the following conditions are equivalent:

1. *The category $\mathscr{X}$ is a topos; hence equivalent to the category of* $\mathrm{Shv}(\boldsymbol{C})$ *for* $\boldsymbol{C}$ *a small category with a Grothendieck topology.*
2. *The category $\mathscr{X}$ is a left exact localization of* $\mathrm{PSh}(\boldsymbol{C})$.
3. *Giraud's Axioms are satisfied:*
   - i. *Equivalence relations in $\mathscr{X}$ are effective.*
   - ii. *Coproducts in $\mathscr{X}$ are disjoint (and $\mathscr{X}$ admits small coproducts).*
   - iii. *The collection of effective epimorphisms in $\mathscr{X}$ is closed under pullback.*
   - iv. *The formation of coproducts commutes with pullback: that is, for every morphism $f : X \to Y$ in $\mathscr{X}$, the pullback functor $f^* : \mathscr{X}_{/Y} \to \mathscr{X}_{/X}$ preserves coproducts.*
   - v. *There exists a set of objects $\mathscr{U}$ of $\mathscr{X}$ which generate $\mathscr{X}$ as:* $\forall X \in \mathrm{obj}(\mathscr{X})$ *there exists covering* $\{U_i \to X\}$, *where* $U_i \in \mathscr{U}$.

Reframing The Implication

We will prove a slightly modified statement, with the additional assumption that $\mathscr{X}$ is closed under finite limits.

Theorem (❸′ ⇒ ❶′)

Let $\mathscr{X}$ be a category satisfying Giraud's Axioms and $\mathbf{C} \subseteq \mathscr{X}$ be a full subcategory of $\mathscr{X}$ which is closed under finite limits and generates $\mathscr{X}$ (in the sense of ⓥ). Say that a family of morphisms $\{U_i \to X\}$ in $\mathbf{C}$ is a covering if is a covering in $\mathscr{X}$ (as defined above). Then:

- ⓐ *The collection of covering families determines a Grothendieck topology on $\mathbf{C}$.*
- ⓑ *For every object $Y \in \mathrm{obj}(\mathscr{X})$ let $h_Y : \mathbf{C}^{op} \to Set$ denote the functor represented by Y on the subcategory $\mathbf{C}$, given by $h_Y(X) = Hom_{\mathscr{X}}(X, Y)$. Then h_Y is a sheaf with respect to the Grothendieck topology above.*
- ⓒ *The construction $Y \mapsto h_Y$ induces an equivalence of categories $h : X \to \mathrm{Shv}(\mathbf{C})$.*

Reframing The Implication

We then note that this ❸′ implies ❶′ (the statement that $\mathscr{X}$ is a topos with finite limits) then ❶′ vacuously implies our original ❶ and thus if we show ❸′ $\Rightarrow$ ❶′ we conclude our proof.

A Remark on the Construction of h

Note using the inclusion of full sub-category $\mathbf{C}$ of $\mathscr{X}$ and the fact that $\mathrm{PSh}(\mathbf{C})$, $\mathscr{X}$ are presentable we obtain the diagram:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\quad i \quad} & \\ \big\downarrow & & \searrow \\ \mathrm{PSh}(\mathbf{C}) & \underset{i^*}{\overset{i_!}{\rightleftarrows}} \ (\perp) & \mathscr{X} \end{array}$$

Where for any $X \in \mathrm{obj}(\mathscr{X})$ we have $i^*(X) : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$ defined as $\mathrm{obj}(\mathbf{C}^{\mathrm{op}}) \ni D \mapsto \mathrm{Hom}_{\mathscr{X}}(i(D), X) = h_X \circ i$, for the representable presheaf h_X from the Yoneda Lemma.

A Remark on the Construction of h (Cont.)

Then $h_X \circ i$ is a sheaf iff we have h as in the diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{i^*} & \mathrm{PSh}(\mathbf{C}) \\ & \overset{h}{\searrow} & \big\uparrow \\ & & \mathrm{Shv}(\mathbf{C}). \end{array}$$

So then this h defined as $\mathrm{obj}(X) \ni Y \mapsto h_Y \circ i$ is the precise construction of the functor in our theorem, whose image we write more simply as just h_Y.

Proof of Theorem [(3′) ⇒ (1′)]

Proof.

ⓐ Consider $\{U_i \to X\}$ in $\mathbf{C}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$. We wish to show that the collection projection maps $\{U_i \times_X Y \to Y\}$ is also a covering. That is, given that $\coprod U_i \to X$ is an effective epi in $\mathscr{X}$ wts that the induced map $\coprod(U_i \times_X Y) \to Y$ is the same. This is so, as (iii), (iv) assure us that pulling back along f preserves coproducts and being an effective epi.

Suppose we have a coverings $\{U_i \to X\}$ and $\{V_{i,j} \to U_i\}$ in $\mathbf{C}$. Now wts that the composite maps $\{V_{ij} \to X\}$ are also a covering. Let $X_0 \subseteq X$ be a subobject such that each $V_{i,j} \to X$ factors through X_0. Then $V_{i,j} \to U_i$ factors through $X_0 \times_X U_i, \forall i$. As $V_{i,j}$ cover U_i, each of the $U_i \to X$ factors through X_0, i.e. $X_0 \times_X U_i = U_i$. And as $\{U_i \to X\}$ cover X we deduce $X_0 = X$.

Finally, note the collection $\{f_i : U_i \to X\}$ is a covering when f_i admits a section, because then f_i is itself an effective epi in $\mathbf{C}$.

$(3') \Rightarrow (1')$ [Cont.]

Proof.

(b) Fix $Y \in \mathrm{obj}(\mathscr{X})$; wts that $h_Y : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf. Let $\{U_i \to X\}$ be any covering of $X \in \mathrm{obj}(\mathbf{C})$, so wts that

$$h_Y(X) \to \prod h_Y(U_i) \rightrightarrows \prod h_Y(U_i \times_X U_j)$$

is an equalizer. Expanding the above mapping yields:

$$\mathrm{Hom}_{\mathscr{X}}(X, Y) \to \mathrm{Hom}_{\mathscr{X}}(\coprod_i U_i, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{X}}(\coprod_{i,j} U_i \times_X U_j, Y)$$

and as $\{U_i \to X\}$ is a covering it follows that $\coprod_i U_i \to X$ is an effective epi. So we're done as the map below is an iso by (iv).

$$\coprod_{i,j}(U_i \times_X U_j) \to (\coprod_i U_i) \times_X (\coprod_j U_j)$$

A Necessary Lemma

Lemma

Let $\{U_i \to X\}_{i \in I}$ *be a covering in* $\mathscr{X}$*. Then the induced map* $\coprod h_{U_i} \to h_X$ *is an effective epimorphism in* $\mathrm{Shv}(\boldsymbol{C})$.

Proof.

Consider a section $s \in h_X(C), C \in \mathrm{obj}(\mathbf{C})$ given by morphism $C \to X$ in $\mathscr{X}$. As $\{U_i \to X\}$ is a covering the induced map $\coprod U_i \to X$ is an effective epi in $\mathscr{X}$. By (iii), (iv) it follows that $\{U_i \times_X C \to C\}$ is also a covering in $\mathscr{X}$. And as $\mathrm{obj}(\mathbf{C})$ generates $\mathscr{X}$, each $U_i \times_X C$ admits a covering $\{V_{i,j} \to U_i \times_X C\}$ where $V_{i,j} \in \mathrm{obj}(\mathbf{C})$.

Then the composite maps collection $\{V_{i,j} \to C\}$ is a covering in $\mathbf{C}$. By construction $\forall (i,j)$ the image $s_{i,j} \in h_X(V_{i,j})$ of s belongs to the image of $h_{U_i}(V_{i,j}) \to h_X(V_{i,j})$. So, allowing C, s to vary we conclude $\{h_{U_i} \to h_X\}$ is a covering of h_X in $\mathrm{Shv}(\mathbf{C})$. $\square$

A Second Proof of the Lemma

Proof.

We want to show that $\coprod_i h_{U_i} \to h_X$ is an effective epi. To this end we we value the map at some arbitrary $\alpha \in H_X(C), C \in \mathbf{C}$. Then cover $U_i \times_X C$ by some $V_{i,j} \in \mathbf{C}$ from which we can map $\alpha \in h_X(C)$ to $h_X(V_{i,j})$ and then we can pull back to $h_{U_i}(V_{i,j}), \forall i, j$. Thus, since α, C we arbitrary we see that the desired condition on the codomain h_X being a coequalizer and we have an effective epi. $\square$

Proof.

(c) First we wish to show that $h : \mathscr{X} \to \mathrm{Shv}(\mathscr{X})$ is fully faithful. Meaning that

$$\forall X, Y \in \mathrm{obj}(\mathscr{X}), \theta_X : \mathrm{Hom}_{\mathscr{X}}(X, Y) \to \mathrm{Hom}_{\mathrm{Shv}(\mathbf{C})}(h_X, h_Y)$$

is bijective. Fix Y and say that X is *good* if θ_X is bijective. We know proceed in several steps:

- (I) Every $X \in \mathrm{obj}(\mathbf{C})$ is good by the Yoneda Lemma.
- (II) Suppose that $X \in \mathrm{obj}(\mathscr{X})$ admits covering $\{U_i \to X\}$. We claim if each U_i and fiber product $U_i \times_X U_j$ is good then so is X. To show this, note that $\coprod_i U_i \to X, \coprod_i h_{U_i} \to h_X$ are effective epis in $\mathscr{X}, \mathrm{Shv}(\mathbf{C})$, respectively, by the Lemma. And as both categories satisfy (iv) and h preserves finite limits we obtain coequalizer diagrams:

Proof.

II

$$\coprod_{i,j} U_i \times_X U_j \rightrightarrows \coprod_i U_i \longrightarrow X$$

$$\coprod_{i,j} h_{U_i \times_X U_j} \rightrightarrows \coprod_i h_{U_i} \longrightarrow h_X.$$

Thus, θ_X fits in the following commutative diagram of sets:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathscr{X}}(X,Y) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathscr{X}}(U_i,Y) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathscr{X}}(U_i \times_X U_j, Y) \\ \downarrow \theta_X & & \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}_{\mathtt{Shv}(\mathbf{C})}(h_X,h_Y) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathtt{Shv}(\mathbf{C})}(h_{U_i},h_Y) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathtt{Shv}(\mathbf{C})}(h_{U_i \times_X U_j}, h_Y) \end{array}$$

where rows are equalizer diagrams. Thus we conclude θ_X is an iso.

The Final Step: $[(3') \Rightarrow (1')](c)$

Proof.

- III Now let $X \in \mathrm{obj}(\mathbf{C})$, $U \subseteq X$. Then we claim U is good. To prove this, pick a covering $\{U_i \to U\}$ where U_i belongs to $\mathbf{C}$. Since here we assume $\mathbf{C}$ is closed under finite limits each fiber product $U_i \times_U U_j \cong U_i \times_X U_j$ belongs to $\mathbf{C}$ (where we use subobject embedding, essentially pulling back along it). Thus from (I) the objects $U_i, U_i \times_X U_j$ are good and so U is good by (II).
- IV Let $X \in \mathrm{obj}(\mathbf{C})$ and pick covering $\{U_i \to X\}, U_i \in \mathrm{obj}(\mathbf{C})$. Then every fiber product $U_i \times_X U_j$ is a subobject of $U_i \times U_j$ in $\mathbf{C}$ and therefore is good by (III), so we conclude X is good by (II).

Thus $h : \mathscr{X} \to \mathrm{Shv}(\mathscr{X})$ is indeed fully faithful.

The Final Step: $[(3') \Rightarrow (1')](c)$

Proof.

We now wish to show that h is preserves coproducts. We first then apply the Lemma to the case where $I = \emptyset$ and so deduce that h maps the initial object of $\mathscr{X}$ to that in $\mathrm{Shv}(\mathbf{C})$. Now fix $\{X_i\} \in \mathrm{obj}(\mathscr{X})$ with coproduct X, then wts that $\theta : \coprod h_{X_i} \to h_X$ is an iso in $\mathbf{C}$. Our Lemma says tells us that θ is an effective epi and so it suffices to show that θ is too a mono, i.e.

$$\coprod h_{X_i} \xrightarrow{\delta} (\coprod h_{X_i} \times_{h_X} \coprod h_{h_{X_j}})$$

is an iso. Recall now that $\mathrm{Shv}(\mathbf{C})$ satisfies (iv) and h is right exact (finite colimit preserving) so we may rewrite $\mathrm{cod}(\delta) = \coprod_{i,j} h_{X_i \times_X X_j}$. Thus we must show that $h_{X_i} \to h_{X_i \times_X X_j}$ are isos and $h_{X_i \times_X X_j}$ is an initial object of $\mathrm{Shv}(\mathbf{C}), i \neq j$. These follow from that fact that coproducts in $\mathscr{X}$ are disjoint by (iii) and that h preserves monos.

The Final Step: $[(3') \Rightarrow (1')](c)$

Proof.

Now we wish to show that h is essentially surjective. Select $\mathscr{F} \in \mathrm{Shv}(\mathbf{C})$; wts that $\mathscr{F}$ belongs to the essential image of h. First, consider the case where $\mathscr{F} \subseteq h_X$ for some $X \in \mathrm{obj}(\mathscr{X})$ and pick effective epi $\coprod h_{C_i} \to \mathscr{F}, C_i \in \mathbf{C}$. Set $U := \coprod_i C_i$ to obtain an effective epi $h_U \to \mathscr{F}$ for some $U \in \mathscr{X}$ then $h_U \to \mathscr{F} \hookrightarrow h_X$ arises from $u \in \mathrm{Hom}_{\mathscr{X}}(U, X)$ and since $\mathscr{X}$ is a pretopos u factors as $U \xrightarrow{u'} Y \xrightarrow{u''} X$ for effective epi u' and mono u''. (Note for the induced maps: $h_U \xrightarrow{u'} h_Y$ is an effective epi in $\mathrm{Shv}(\mathbf{C})$ by our Lemma and $h_Y \xrightarrow{u''} X$ is a mono in $\mathrm{Shv}(\mathbf{C})$ as h is lex). As images are unique in a pretopos we conclude that $\mathscr{F} \cong h_Y$.

Now suppose that $\mathscr{F}$ is any sheaf on $\mathbf{C}$ and again pick effective epi $h_U \to \mathscr{F}, U \in \mathrm{obj}(X)$.

The Final Step: $[(3') \Rightarrow (1')](c)$

Proof.

In this case, the fiber product $h_U \times_{\mathscr{F}} h_U$ is a sheaf on $\mathbf{C}$ which can be viewed as a subobject of $h_U \times h_U = h_{U \times U}$. By the same logic as before we can pick an iso $h_U \times_{\mathscr{F}} h_U \cong h_R, R \in \mathrm{obj}(\mathscr{X})$. Now consider the canonical bijection

$$\forall Y \in \mathrm{obj}(\mathscr{X}), \mathrm{Hom}_{\mathscr{X}}(Y, R) \cong \mathrm{Hom}_{\mathrm{Shv}(\mathbf{C})}(h_Y, h_u \times_{\mathscr{F}} h_U)$$

so we can view R as an equivalence relation on U in $\mathscr{X}$.

Thus by (i) this equivalence relation is effective. That is, there exists an effective epi $U \to X$ in $\mathscr{X}$ with $R = U \times_X U$ (as subobjects of $U \times U$). Therefore we apply our Lemma to the covering $\{U \to X\}$ and finally obtain the isomorphism:

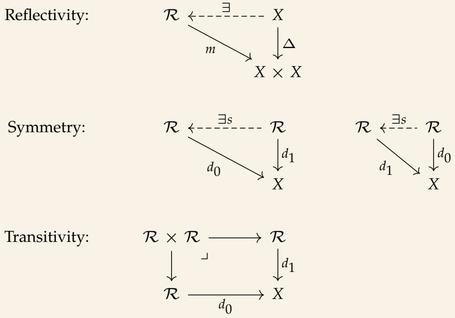
$$h_X \cong \mathrm{Coeq}(h_R \rightrightarrows h_U) \cong \mathrm{Coeq}(h_U \times_{\mathscr{F}} h_U \rightrightarrows h_U) \cong \mathscr{F}. \quad \square$$

Thank You!

Appendix I

Definition

An **Equivalence Relation** is a relation $\mathcal{R}$ of $X \in \mathrm{obj}(\mathscr{X})$ satisfying the following conditions:



And note if $\mathscr{X}$ admits colimits then we could construct the equalizer, q of d_0, d_1.

Appendix II

Definition

A **Pretopos** is a category **C** satisfying the following conditions:

- **C** admits finite limits.
- Every equivalence relation in **C** is effective.
- **C** admits finite coproducts, and coproducts are disjoint.
- The collection of effective epimorphisms in **C** is closed under pullbacks.
- Finite coproducts in **C** are preserved by pullback.

This is to say that **C** is exact (regular and every congruence pair is a kernel) and extensive (coprods work well with pullback).

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