Flatness and filtered colimits, Diaconescu's theorem

1 Conventions

For $D : \mathcal{I} \to \mathcal{C}$, write

$$\mathsf{cone}(D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set} \qquad \mathsf{cocone}(D) : \mathcal{C} \to \mathbf{Set}$$

for the functors of cones and cocones given by $\mathsf{cone}(D)(C) = \mathsf{lim}_i\, \mathcal{C}(C, D_i)$ and $\mathsf{cocone}(D)(C) = \mathsf{lim}_i\, \mathcal{C}(D_i, C)$, and denote by $\mathsf{Cone}(D)$ and $\mathsf{Cocone}(D)$ the categories of cones and cocones given by $\mathsf{Cone}(D) = \mathsf{coelts}(\mathsf{cone}(D))$ and $\mathsf{Cocone}(D) = \mathsf{elts}(\mathsf{cocone}(D))$, respectively.

Note that we have $\mathsf{cone}(D) = \mathsf{lim}(よ \circ D)$ and $\mathsf{cocone}(D) = \mathsf{lim}(Z \circ D^{\mathsf{op}})$.

Just like we write (co)limits sometimes with and sometimes without binders, i.e. $\mathsf{colim}(D) = \mathsf{colim}_i(D_i)$, we sometimes like to write categories of (co)elements sometimes with binders, thinking of them as (op)lax colimits. Specifically we write

$$\mathsf{elts}(D) = \mathsf{elts}_i(D_i) \qquad \mathsf{coelts}(D) = \mathsf{coelts}_i(D_i).$$

We can reconstruct colimits from (op)lax colimits by taking connected components.

$$\pi_0(\mathsf{elts}_i(D_i)) = \underset{i}{\mathsf{colim}}(D_i) \qquad \pi_0(\mathsf{coelts}_i(D_i)) = \underset{i}{\mathsf{colim}}(D_i)$$

2 The proofs

Definition 1 A small category $\mathcal{F}$ is called *filtered*, if $\mathcal{F}$-colimits commute with finite limits. ♢

Lemma 2 *$\mathcal{F}$ is filtered iff $\mathsf{Cocone}(D)$ is connected for all finite diagrams $D : \mathcal{I} \to \mathcal{F}$ iff $\mathsf{Cocone}(D)$ is inhabited for all finite diagrams $D : \mathcal{I} \to \mathcal{F}$.*

Proof. Assume that $\mathcal{F}$ is filtered. We have $\mathsf{cocone}(D)(A) = \mathsf{lim}_i\, \mathcal{F}(D_i, A)$ and $\mathsf{Cocone}(D) = \mathsf{elts}_A(\mathsf{lim}_i\, \mathcal{F}(D_i, A))$. Thus we have

$$\begin{aligned}\pi_0(\mathsf{Cocone}(D)) &= \pi_0(\mathsf{elts}(\mathsf{cocone}(D))) \\ &= \pi_0(\mathsf{elts}(\underset{i}{\mathsf{lim}}\, \mathcal{F}(D_i, A))) \\ &= \underset{A}{\mathsf{colim}}(\underset{i}{\mathsf{lim}}\, \mathcal{F}(D_i, A)) \\ &= \underset{i}{\mathsf{lim}}(\underset{A}{\mathsf{colim}}\, \mathcal{F}(D_i, A)) \\ &= \underset{i}{\mathsf{lim}}\, 1 = 1\end{aligned}$$

i.e. the category of cocones is connected and in particular inhabited.

Conversely, assume that $\mathsf{Cocone}(D)$ is inhabited for all finite diagrams $D : \mathcal{I} \to \mathcal{F}$ and let $P : \mathcal{I} \times \mathcal{F} \to \mathbf{Set}$. We have to show that the canonical comparison arrow

$$\underset{A}{\mathsf{colim}}\, \underset{I}{\mathsf{lim}}\, P(I, A) \to \underset{I}{\mathsf{lim}}\, \underset{A}{\mathsf{colim}}\, P(I, A)$$

is a bijection. On the left-hand side we have equivalence classes of pairs $(A, \alpha : \Gamma(P(-, A)))$ of an object $A \in \mathcal{F}$ and a section α of the diagram $P(-, A) : \mathcal{I} \to \mathbf{Set}$, where two such pair (A, α) and (B, β) are identified if there exists a cospan

$A \xrightarrow{f} C \xleftarrow{f} B$ such that $f_*(\alpha) = g_*(\beta)$. On the right-hand side we form the quotient over each I and thus have sections of equivalence classes instead of equivalence classes of sections. To show that the mapping is injective we have to show that if two sections are pointwise equivalent then they are uniformly equivalent. For this we use a filteredness argument, and so on (fairly combinatorial argument – can we reduce it to specific diagram sizes? Interestingly we don't only use the shape of $\mathcal{I}$ as finite diagram in the argument, but also derived diagrams.)

(The implication 3 to 1 is [Bor94, Theoerm 2.13.4] and [Mac98, pg 215].) ∎

Lemma 3 *For $\mathcal{C}$ a small finite-limit category,* $\mathbf{Lex}(\mathcal{C}, \mathbf{Set})$ *is closed under filtered colimits in* $[\mathcal{C}, \mathbf{Set}]$*.*

Proof. Let $D : \mathcal{F} \to \mathbf{Lex}(\mathcal{C}, \mathbf{Set})$ be a filtered diagram. We have to show that the pointwise colimit $P = \mathsf{colim}_{A\in\mathcal{F}} DA : \mathcal{C} \to \mathbf{Set}$ preserves finite limits. Let $E : \mathcal{I} \to \mathcal{C}$ be a finite diagram. We have

$$\begin{aligned} P(\lim_i E_i) &= \operatorname*{colim}_{A\in\mathcal{F}} DA(\lim_i E_i) \\ &= \operatorname*{colim}_{A\in\mathcal{F}} \lim_i DA(E_i) \\ &= \lim_i \operatorname*{colim}_{A\in\mathcal{F}} DA(E_i) \\ &= \lim_i P(E_i) \end{aligned}$$

Theorem 4 *[ABLR02, Theorem 2.4] TFAE for a small diagram $F : \mathbb{C} \to \mathbf{Set}$:*

(i) *F^* preserves finite limits of representables;,*

(ii) *F^* preserves finite limits;*

(iii) *F is a filtered colimit of representables;*

(iv) $\mathsf{elts}(F)$ *is cofiltered;*

(v) *(if $\mathbb{C}$ is lex) F preserves finite limits.*

Proof. (iv) ⇒ (iii) and (ii) ⇒ (i) are obvious.

For (iii) ⇒ (ii) observe that if F is a filtered colimit of representables, then F^* is a filtered colimit of evaluation functors.

For (i) ⇒ (iv) assume that F^* preserves finite limits of representables and let $D : \mathcal{I} \to \mathsf{elts}(F)$ be a finite diagram. We have to show that $\mathsf{Cone}(D)$ is connected, and we do so indirectly by considering cones on UD, where $U : \mathsf{elts}(F) \to \mathbb{C}$ is the forgetful functor. We have

$$\begin{aligned} \int^C FC \times \mathsf{cone}(UD)(C) &= F^*(\mathsf{cone}(UD)) \\ &= F^*(\lim(\text{よ} UD)) \\ &= \lim(F^* \text{よ} UD) \\ &= \lim(FUD) \end{aligned}$$

On the LHS we have equivalence classes of triples (C, x, κ) where $C \in \mathbb{C}$, $x \in F(C)$ and κ is a cone between C and UD on the RHS we have sections of FUD which can be viewed as diagrams in $\mathsf{elts}(D)$ over UD. The computation shows that the construction transforming pairs of a C-cone and an x over C to a diagram over UD is a bijection. In particular, there is an inverse image of the diagram D, which gives a cone over D. ∎

Lemma 5 *Flat presheaves are closed under filtered colimits in $\widehat{\mathbb{C}}$.*

Proof. Let $D : \mathcal{F} \to \widehat{\mathbb{C}}$ be a filtered of flat presheaves. It is enough to show that $\mathsf{Lan}_Z\,\mathsf{colim}_A\,PA$ preserves finite limits. We have

$$\mathsf{Lan}_Z(\operatorname*{\mathsf{colim}}_A(PA)) = \operatorname*{\mathsf{colim}}_A(\mathsf{Lan}_Z(PA))$$

and on the right we have a filtered colimit of lex functors. ∎

3 Torsors in the sense of Johnstone

If $\mathbb{C}$ is a small category and $\mathcal{E}$ is a topos, functors $F : \mathbb{C}^{\mathsf{op}} \to \mathcal{E}$ correspond to fibered functors $[F] : [\mathbb{C}^{\mathsf{op}}] \to [\mathcal{E}] \simeq \Delta^* P_{\mathcal{E}}$ which correspond to functors $[\Delta\mathbb{C}^{\mathsf{op}}] \simeq \Delta_![\mathbb{C}^{\mathsf{op}}] \to P_{\mathcal{E}}$ by transposition. The latter correspond to discrete fibrations $\mathbb{F} \to \Delta\mathbb{C}$ in $\mathcal{E}$. We call F a *torsor* if $\mathbb{F}$ is filtered in $\mathcal{E}$.

Lemma 6 *$Z : \mathbb{C}^{\mathsf{op}} \to [\mathbb{C}, \mathbf{Set}]$ is a torsor.*

Proof. Kind of clear since the pointwise fibers have terminal objects. ∎

The discrete fibration in $[\mathbb{C}, \mathbf{Set}]$ corresponding to Z is 'generic' in a sense (of classifying toposes). Its fibers are probably $C/\mathbb{C} \to \mathbb{C}$. Applying an algebraic functor $F^* : [\mathbb{C}, \mathbf{Set}] \to \mathbf{Set}$ yields a discrete fibration whose codomain is equivalent to $\mathbb{C}$, and one can show that it corresponds to $F : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$. This is how Johnstone shows that F is flat whenever F^* is lex.

References

[ABLR02] J. Adámek, F. Borceux, S. Lack, and J. Rosický. A classification of accessible categories. *Journal of Pure and Applied Algebra*, 175(1-3):7–30, 2002.

[Bor94] F. Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994.

[Mac98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.